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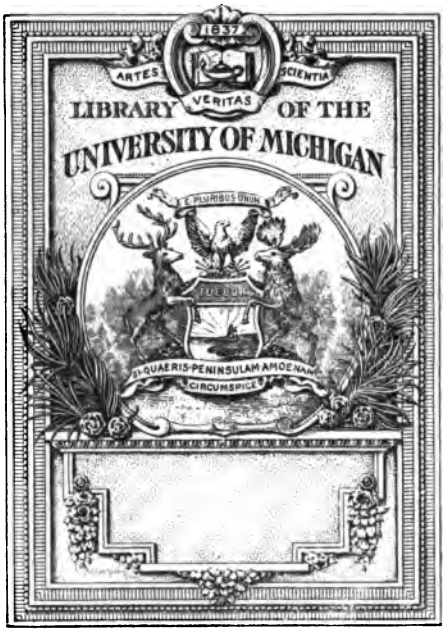
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TRUE DOCTRINE OF ORBITS,

AN ORIGINAL

TREATISE ON CENTRAL FORCES.

By H. G. RUSH,
OF NEW DANVILLE, PENN'A.

"I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me." —SIR ISAAC NEWTON.

LANCASTER, PA.
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PREFACE.

THE creation of this little book is owing to a chain of those fortuitous circumstances to which science and art are indebted for so many of their treasures.

The treasures of art and the baubles of folly, like a wise man and a fool, may readily be mistaken for each other; and it is difficult to devise whether the wise are wise and fools fools, because those of assumed competence pronounce them so.

Agassiz has said, that the first remark about a scientific discovery which conflicts with existing notions is "that it is not true"; next it is branded as "anti-Christian"; and finally, every body "knew it long ago."

In the face of existing prejudices, the author has vividly pictured to himself the ridicule which will be his portion if he fails to establish his propositions. Were he to come in conflict with authority so well fathered, and so long, and so ably sustained, in any other than an exact science, nerve would fail him to face the reproaches and criticisms of professionals. Confident, however, that the correctness of his conclusions must soon be admitted, that his investigations will popularize, simplify, and enlarge one of the most interesting and valuable departments open to mathematical research, he feels assured that there is but little need to apologize for the intrusion of this unwelcome novelty.

An original investigation of the "problem of lights" conducted by the author some seventeen years ago resulted in a series of conclusions which, together with certain inconsistencies associated with the elliptical theory, first suggested the "true doctrine of orbits."

For want of time and courage, (as the task appeared too presumptuous in his youth), the problem was dismissed from his mind and remained closeted while the engagements of an active farm life occupied his time. Meanwhile, the author's dissatisfaction

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with the elliptical theory kept alive the desire "some day" to continue his investigations. In the winter of 1885-86, a period of convalescence afforded an opportunity to review and extend his work —somewhat at a disadvantage for want of exercise. The humble circumstances under which this work was executed is offered as an apology for its appearing less attractively and less logically than the author would desire. Herein he asks for charity, desiring none for the incorrectness of his theory or any of his conclusions.

In the developments of the True Doctrine of Orbits, the author has used no debatable premises, unless we choose to question the correctness of the law of gravitation, the verifications of which are so commendable to the understanding as to excite no dispute. This same law which furnishes the fundamental proposition in the new theory is also of vital import to the old.

The objection raised by our confident critics "that the existing theory of orbits so well satisfies astronomical phenomena as to leave no doubt of its correctness" is but a repetition of the obstacles which confronted every revolutionary scientific movement.

The system of the Astrologer of ancient times was all sufficient in its day. The advocates of Ptolemy thought they had accounted for all the puzzles open to their view.

The true lover of science takes no delight in the perpetuation of error however ably supported, nor will he refuse to hear a better solution. The incontestable truths of science, (of which exact science alone can boast) are, many of them, developed by gradation from utter darkness to noon-day light; then why should we stand aghast to hear of another step of advance in the direction of "the true."

That Newton erred in his conclusion, framed as a preconception and sustained by ingenious demonstration, need not be greatly wondered at, nor that the talent of the ripest centuries was satisfied to account him correct; but that any one should persistently refuse to see when new light beams full in the face is unworthy of a better name than bigot. Newton himself would take no delight in the perpetuation of the errors which find protection in his garner of truth. He would not hesitate to acknowledge the finer symmetry of a newly found pebble, though picked up by other hands. He was not proud of his discoveries, knowing that if he saw

farther than his predecessors it was because he had their shoulders to stand upon; and has he scanned the horizon to so great a depth that none other may gain advantage by perching upon his lofty brow? But I am growing too presumptuous.

Kind reader, we ask of you an unbiased perusal of this work, knowing that no prefatory remarks of defense can possibly help to impress you otherwise than unfavorably.

H. G. RUSH.

New Danville, Jan. 1887.

NOTICE.

IT was found after this work had gone partly through the press that an indiscretion had been committed in a minor proposition, the correction of which is found in the last article of the appendix. The error referred to is in Theorem VII, and some of its effect is transmitted to Theorems XII and XIV, of which the reader please make note at once to save confusion.

In Theorem IX the words "of equal eccentricity" should be omitted, since the demonstration is general, independent of eccentricity.

Should any find exception to our criticism on Newton's demonstration of force as related to the ellipse, we ask them to consult lemma 1 of the Principia and find that the equality of Qx and Qv is not warranted. He says:—"*Quantities, and the ratios of quantities, which, in any finite time, tend constantly to equality, and which, before the end of that time, approach nearer to each other than by any assigned difference, become ultimately equal.* The quantities of which Newton asserts equality are not of the kind here referred to.

The reason for confining our comments mainly to Newton is that we have not succeeded in securing a copy of the demonstrations by La Place.

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THE

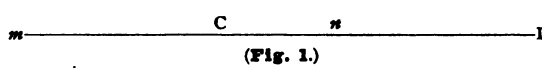
TRUE THEORY OF ORBITS.

SECTION I.

Development by a System of Double Co-ordinates of the Curve Representing the Equality of Forces Emanating from Two Distinct Centres.

The circular theory of orbits having originated in a general discussion of the *problem* of *lights*, which applies equally to light, heat, electricity, gravitation and all other forces which act in degree, inversely as the square of the distance, the initial propositions will be given as developed seventeen years ago.

Required to find any point equally acted upon by two powers, whether light, heat, electricity, gravitation, etc., when their positions and relative intensities are given.



(Fig. 1.)

Let C and D be the positions of the given forces.

a = the distance between them.

b = intensity of D.

c = intensity of C.

x = distance from C in the line CD, at which the forces act equally. C being the origin of co-ordinates, distance to the right is regarded positive and to the left negative.

From principals in optics we derive

$$\frac{b}{(a-x)^2}=\frac{c}{x^2}, \text{ solving which}$$

$$x=\frac{-a(c\pm\sqrt{bc})}{b-c}, \text{ assuming } b>c, \text{ one value of } x \text{ is}$$

positive but the other value is negative and greater than the positive. These results show that there are two points on *opposite* sides of the force *assumed* the lesser, where the given forces act equally.

$a-x$ being the distance from D at which the forces are equal, its value is less than a when x is positive and greater than a when x is negative.

$$a-x=a-\frac{-a(c\pm\sqrt{bc})}{b-c}=\frac{a(b\pm\sqrt{bc})}{b-c}$$

a refers to D as its origin, and may itself be regarded positive or negative as it is placed to the right or left of C.

When a is positive *both* values of $a-x$ in the preceding expression are *positive*, and when a is *negative* both values of $a-x$ are *negative*, showing that both the points of equal activity lie on the *same* side of the force assumed the greater.

Having thus far assumed $b>c$, let us now assume $b<c$, then in

$$x=\frac{-a(c\pm\sqrt{bc})}{b-c}, \text{ one of the values is clearly positive and}$$

the other is positive because $c>\sqrt{bc}$.

And in $a-x=\dfrac{a(b\pm\sqrt{bc})}{b-c}$, one and the greater value is negative while the other and lesser value is positive.

Thus we see that whatever the supposition with reference to the relative intensities of the forces, the points of equal activity will fall on opposite sides of the force *assumed* the lesser, and on the *same* side of the greater.

When $b=c$, $x=\dfrac{-a(c\pm\sqrt{bc})}{b-c}=\infty$, or $-(a-x)$ and

$$a-x=\frac{a(b\pm\sqrt{bc})}{b-c}=\infty \text{ or } -x;$$

that is, at only one finite point in the line $C\,D$ will the *effect* of the forces be equal when the forces themselves are equal.

When $a = o$ and $b >^{\text{or}} < c$, $x = o$ and $a - x = o$; that is, at no point will the forces act with equal intensities.

When $a = o$ and $b = c$.

$x = \frac{o}{o}$ and $a - x = \frac{o}{o}$, showing that in this case all points are equally acted upon.

For the point of equal action which lies between the forces,

$$x = \frac{-a(c - \sqrt{bc})}{b - c}, \quad a - x = \frac{a(b - \sqrt{bc})}{b - c}.$$

Since the denominators of these fractions are equal, their relation depends on the numerator. Having assumed $b > c$ the numerical value of $x < a - x$, $2x < a$, $x < \frac{a}{2}$, that is, the point of equal action is nearer the lesser than the greater force.

It is evident that not only in the two points already found will the given forces act with equal intensity, but that an envelope of points surrounds the weaker force. In any one plane those points will determine a curve connecting the points m and n of Fig. 1.

It is required to find a general equation for this curve, its nature and properties.

THEOREM I.

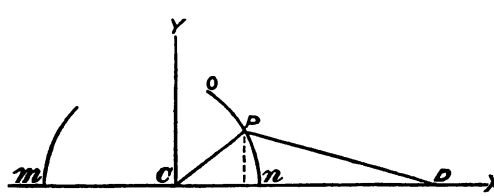


Fig. 2.

The equation of the curve of equal activity surrounding the lesser force is $x^2 + y^2 = \frac{ac(a - 2x)}{b - c}$.

a, b and c represent the same values as in Fig. 1, while x and y are the general co-ordinates of the curve referred to rectangular axes, origin at the lesser force.

We will assume that $n\ o$ is a part of the curve in question, of which P is any point.

Let r represent the radius vector $C\,P$,
and r' equal the radius vector $P\,D$,
then since the forces exerted at P are equal, being *directly* as the masses of C and D and *inversely* as the square of their distances.

$$\frac{b}{r'^2} = \frac{c}{r^2},$$

But $r'^2 = (a-x)^2 + y^2$ and $r^2 = x^2 + y^2$.

Hence $\dfrac{b}{(a-x)^2 + y^2} = \dfrac{c}{x^2 + y^2}$,

$$bx^2 + by^2 = a^2c - 2acx + cx^2 + cy^2.$$

$$y^2(b-c) + x^2(b-c) = ac(a-2x).$$

$x^2 + y^2 = \dfrac{ac}{b-c}(a-2x)$, which is the equation of the curve.

Observations. $x^2 + y^2 = r^2$.

Hence $r^2 = \dfrac{ac}{b-c}(a-2x)$ that is, the radius vector from C is a variable depending on x, and is $= > < \dfrac{a^2c}{b-c}$ as x is $o + -$ respectively.

Taking the expression $r^2 = \dfrac{ac}{b-c}(a-2x)$ and assigning numerical values to a, b and c, any number of points in the curve may be found by variations of x, by the aid of which the entire curve may be constructed.

Suppose in $r^2 = \dfrac{ac}{b-c}(a-2x)$, $b = c$,

then $r^2 = \dfrac{ac}{o}(o) = \dfrac{o}{o}$ since $2x = a$ when the forces are equal. This expression indicates a general indetermination of r, but observing that $r^2 = x^2 + y^2$, the value of r^2 can never be less than x^2, r not less than $\pm x$, $\therefore$ not less than the least value of x. From these statements we see that the minimum of $r = x = \dfrac{a}{2}$ beyond which it is indeterminate.

The interpretation of this case is, that when the forces are equal, the points of equal activity will constitute the right line bisecting a at right angles.

In $r^2 = \frac{ac}{b-c}(a-2x)$, suppose $b < c$ and the difference S,

then $r = \sqrt{\frac{ac(a-2x)}{-S}}$, which is an imaginary quantity, showing the introduction of an inconsistency. The inconsistency is the assumption that the curve of equal activity surrounds the greater force.

Supposing $b > c$ in $r^2 = \frac{ac}{b-c}(a-2x)$

then, since r^2 is positive for all values of x; $a > 2x$ when x is positive, but may not be greater when x is negative. It may also be observed that the value of r^2 is equally satisfied whether a be positive or negative. The supposition that $b > c$ is the *only* one which is satisfactory and their relation must at all times be such in the general equation.

In $r^2 = \frac{ac}{b-c}(a-2x.)$ Let $x = \frac{a}{2}$,

then, as shown before, $b = c$ and $r = \frac{o}{o}$, that is, indeterminate beyond its minimum value $\frac{a}{2}$,

Let $b = c$ and $a = o$, then $r = \frac{o}{o}$, that is, r is intermediate without any reference to x. The interpretation of this form is. that when the forces are equal and in the same place, all points in space are equally effected by them.

Let $a = o$ and $b > c$ and their difference S, then $r = \frac{o}{\sqrt{S}} = o$; i. e. when the forces are together and of unequal power, they will at no point exercise equal influences.

$$r^2 = x^2 + y^2$$

$$r'^2 = (a-x)^2 + y^2$$

$$r'^2 - r^2 = a^2 - 2ax.$$

$r' + r : a :: a - 2x : r' - r$, or $r' + r : a :: (a-x) - x : r' - r$.

Which expresses that the sum of the raidius vectors to any point of the curve is to the distance between the forces as the difference of the distances from the foot of the ordinate to the forces is to the difference of the radius vectors.

In the general form $x^2 + y^2 = \frac{ac(a-2x)}{b-c}$.

Let $x = o$,

then $y = \pm \frac{a\sqrt{c}}{\sqrt{b-c}}$, the distance at which the curve cuts the axis of ordinates. The positive and negative values being equal, the curve is symmetrical with reference to X.

But when $y = o$, $x = \frac{-ac}{b-c} \pm \frac{a\sqrt{bc}}{b-c}$, the distance at which the curve cuts the axis of abscissas. The inequality of the positive and negative values of x show that the curve is not symmetrical with reference to Y.

THEOREM II.

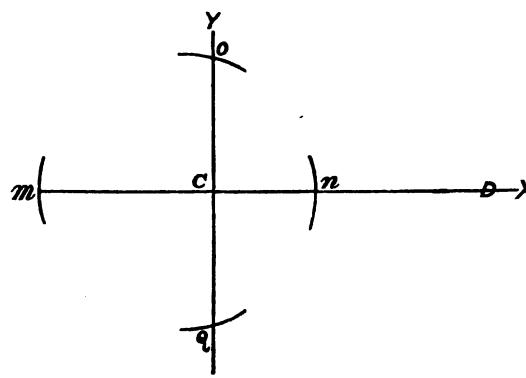


Fig. 3.

The square of the distance at which the curve of equal intensity cuts the axis of abscissas is to the square of the distance on the axis of ordinates, as the intensity of the greater force is to the difference of the forces.

The general equation of the curve is

$$x^2 + y^2 = \frac{ac}{b-c}(a - 2x).$$

We have found that when $x = o$, $y = \pm \frac{a\sqrt{c}}{\sqrt{b-c}}$.

The *numerical* of the positive and negative values of y is therefore $\frac{2a\sqrt{c}}{\sqrt{b-c}}$ which is the distance between the points on the axis of ordinates at which the curve crosses.

We have also found that when $y=o$, $x=\frac{-ac\pm a\sqrt{bc}}{b-c}$. The numerical or geometric value of the positive and negative interpretations of this form taken together is $\frac{2a\sqrt{bc}}{b-c}$, which is the distance between the points on X at which the curve crosses.

Designatiug $\frac{2a\sqrt{bc}}{b-c}$ by A,

and $\frac{2a\sqrt{c}}{\sqrt{b-c}}$ by B,

we derive $\frac{A}{B}=\frac{2a\sqrt{bc}}{b-c}\div\frac{2a\sqrt{c}}{\sqrt{b-c}}=\sqrt{\frac{b}{b-c}}$,

$$\frac{A^2}{B^2}=\frac{b}{b-c},$$

$A^2 : B^2 :: b : b-c$, which is the proposition.

Since $B^2=\frac{a^2c}{b-c}$, $\frac{ac}{b-c}=\frac{B^2}{a}$.

Substituting this value of $\frac{B^2}{a}$ in the general equation we have,

$$x^2+y^2=B^2-\frac{2B^2x}{a}=r^2.$$

$r^2=B^2$ increased or diminished by $\frac{2B^2x}{a}$; diminished, when x is positive and increased when x is negative.

NOTE.—For want of more appropriate terms A and B will be called the major and minor axis of the curve respectively.

THEOREM III.

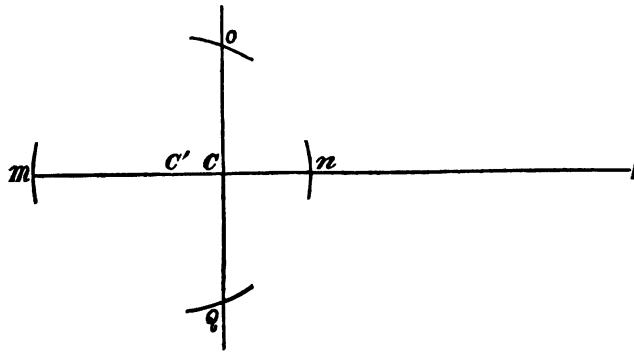


Fig. 4.

The distance from the lesser force to the middle of the major axis is equal to the intensity of the lesser force into the distance between the forces divided by the difference of intensities.

$$x = \frac{-ac \pm a\sqrt{bc}}{b-c}$$

$$\left.\begin{array}{l} \text{Numerically } Cn = -ac + a\sqrt{bc} \\ \text{Numerically } Cm = \quad ac + a\sqrt{bc} \end{array}\right\} \div b - c$$

$Cm - Cn = \frac{2ac}{b-c}$, hence if $\frac{ac}{b-c}$ be taken from Cm and added to Cn their values will be equal $\therefore$ if $CC' = \frac{ac}{b-c}$, $mC' = nC'$ and C' is the middle point of mn or the major axis, which is the proposition.

The value of CC' depends on the values of a, b and c, and gives the curve its character. Call CC' the *eccentricity* of the curve and represent it by e.

Substituting this value in the general equation of the curves it becomes $x^2 + y^2 = e(a - 2x)$, which is a more concise expression and will be used in course of discussion.

THEOREM IV.

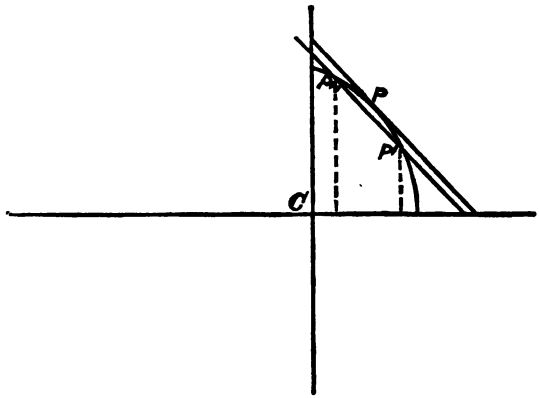


Fig. 5.

The equation of the tangent line to the curve of equal activity is,

$$yy' + xx' = e(a - x - x').$$

$y - y' = \frac{y' - y''}{x' - x''}(x - x')$ is the general equation of a straight line passing through two points.

Let x' and y' be the co-ordinates of the point P', x'' and y'' of P''; then, since these points are in the curve their co-ordinates must satisfy the general equation of the curve.

Hence $y'^2 + x'^2 = e(a - 2x')$,

$$y''^2 + x''^2 = e(a - 2x''),$$

$$y'^2 - y''^2 + x'^2 - x''^2 = -2e(x' - x'')$$

$\frac{y' - y''}{x' - x''} = \frac{-2e - (x' + x'')}{y' + y''}$. But when P' and P'' coincide at P, $x' = x''$ and $y' = y''$, hence

$\frac{y' - y''}{x' - x''} = \frac{-e - x'}{y'}$ = the tangent of the angle of the tangent line with the axis of abscissas.

Substituting $\frac{-e - x'}{y'}$ in the general equation of straight line we have

$$y - y' = \frac{-e - x'}{y'}(x - x')$$

$$yy' - y'^2 = -ex + ex' - xx' + x'^2$$
$$yy' + xx' - y'^2 - x'^2 = e(x' - x)$$
$$y'^2 + x'^2 = e(a - 2x')$$
$$yy' + xx' = e(a - x - x') \text{ the proposition.}$$

In the tangent equation let $x = o$, then $yy' = e(a - x')$.

$y = \frac{e}{y'}(a - x') =$ the distance at which the tangent line cuts the axis of Y.

Since the second member of the equation $yy' = e(a - x')$ is positive, y and y' must be either both positive or both negative.

Let $y = o$ in the tangent equation, then $xx' = e(a - x - x')$.

$$x(x' + e) = e(a - x').$$

$x = \frac{e(a - x')}{x' + e} =$ distance at which the tangent line cuts the axis of X. In this case also x and x' must be of like sign.

$$\frac{e(a - x')}{x' + e} - x' = \frac{e(a - 2x') - x'^2}{x' + e} = \text{sub tangent.}$$

Since $e(a - 2x') = x'^2 + y'^2$,

$$\text{Sub. tan.} = \frac{x'^2 + y'^2 - x'^2}{x' + e} = \frac{y'^2}{x' + e}.$$

The tangent of the angle of the tangent line with the abscissa $\frac{-e - x'}{y'}$ is negative when x is positive or negative and $< e$.

o when x is negative and $= e$.

Positive when x is negative and $> e$.

These properties show that the curve reaches its maximum ordinate when $x = -e$.

THEOREM V.

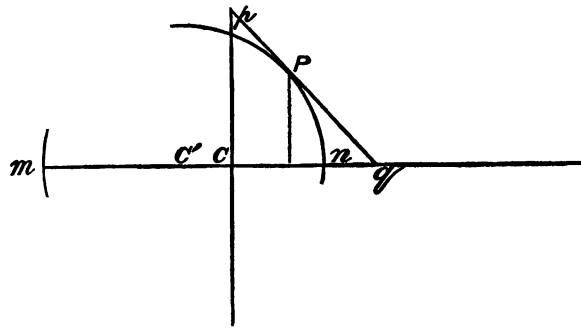


Fig. 6.

The distance from the origin to the intersection of the tangent line and Y, is to the distance from the origin to the intersection of the tangent line and X, as the abscissa of the point of tangency, plus the eccentricity of the curve is to the ordinate of the point of tangency.

Let $s =$ distance on $X = \frac{e(a - x')}{e + x'}$. THEO. IV.

$t =$ distance on $Y = \frac{e(a - x')}{y'}$. THEO. IV.

$\frac{t}{s} = \frac{e + x'}{y'}$

$t : s :: e + x' : y'$, the proposition.

$t > s$ when $e + x' > y'$;

$t < s$ when $e + x' < y'$;

$t = s$ when $e + x' = y'$, that is, when $\frac{e + x}{y'} = 1$.

Now, when $\frac{e + x'}{y'} = 1$, the tangent of the angle of the tangent line with $X = \frac{-e - x}{y} = -1$, hence the angle of the tangent line with x, under the supposition that $e + x' = y'$, is $135°$.

By trigonometry, $tan.\ a, = \frac{sin.\ a.}{cos.\ a.}$

Since $\frac{e + x'}{y'} = 1$, the fraction may be inverted without changing its value.

Assume that $y' =$ sin. and $e + x'$ cos. of the point of contact.

Then $\frac{y'}{e+x'} =$ tan. $= 1$. In this property the curve of equal intensity complies with the requirements of circular functions.

The equations developed by this proposition will be further discussed after the next theorem.

THEOREM VI.

The equation of the normal line to the curve of equal intensity is

$$y - y' = \frac{y'}{e + x'}(x - x').$$

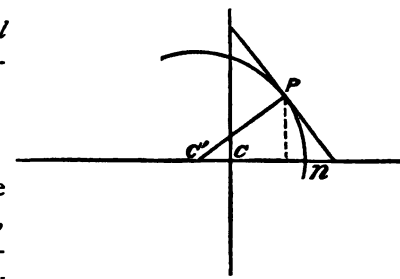


Fig. 7.

If $a =$ tangent of the angle of a tangent line to the curve, and a' the tangent of the angle of the normal line, then by reason of their relation $a = -\frac{1}{a'}$.

But in the curve $a = \frac{-e - x'}{y'}$,

hence $a' = \frac{y'}{e + x'}$.

The general equation of a straight line passing through a given point is $y - y' = a(x - x')$, in which x' and y' are the co-ordinates of the given point and a the tangent of the angle of the line with the axis of X, hence by substitution, the equation of the normal line is $y - y' = \frac{y'}{e + x'}(x - x')$, which is the proposition.

If in the equation $y - y' = \frac{y'}{e + x}(x - x')$, $y = o$.

$$-y' = \frac{y'}{e + x'}(x - x'),$$

$$-x'y' - ey' = xy' - x'y',$$

$x = -e$, which is the point at which the normal line cuts the axis of X. Since x is a constant, every normal line passes through the point $-e$. This property fully reveals the nature of

the curve which may now be constructed in entirety. *The curve of equal intensity is a circle whose centre is at* $-e$.

In the normal equation $y - y' = \frac{y'}{e + x'}(x - x')$.

Let $x = o$ then $y - y' = -\frac{x'y'}{e + x'}$,

$$ey - ey' + x'y - x'y' = -x'y',$$
$$y(e + x') = ey',$$

$y = \frac{ey'}{e + x'}$, which is the distance from the origin at which the normal line cuts the axis of Y.

y is pos. when x' is pos. or neg. and $< e$.

y is neg. when x' is neg. $> e$.

$y = \infty$ when $x = -e$, all of which conclusions are easily verified and in strict accord with the requirements of the circle.

Resuming the tangent equation, Th. IV,

$$yy' + xx' = e(a - x - x').$$

When $x = o$, $y = \frac{e}{y'}(a - x')$, the distance at which the tangent line cuts Y.

Assume further what is possible that $x' = o$, then y and y' are identical and each $= B$, from which (assuming semi-minor axis $= B$)

$$B = \frac{ea}{B},\ B^2 = ea.$$

$e : B :: B : a$, that is, the eccentricity of the curve is to the minor axis as the minor axis is to the distance between the forces. From this proposition we see that a circle whose diameter is ea will include the extremities of the minor axis of the curve of equal intensity in its circumference.

When y' in the equation $y = \frac{e}{y'}(a - x'), = o$

$y = \infty$, in which case x is minimum.

It has been determined that when $y = o$ in the tangent equation

$$x = \frac{e(a - x')}{x' + e}$$

When x' in this form $= -e$, $x = \infty$

when $x' = o$, $x = a$, that is, the line which is the tangent to the curve at the extremity of the minor axis, cuts the axis of abscissas

at a. From the developments thus far obtained it is now possible, when a and e are given, to construct the curve of equal intensity. The following diagram embraces many of the properties resulting from the preceding demonstrations.

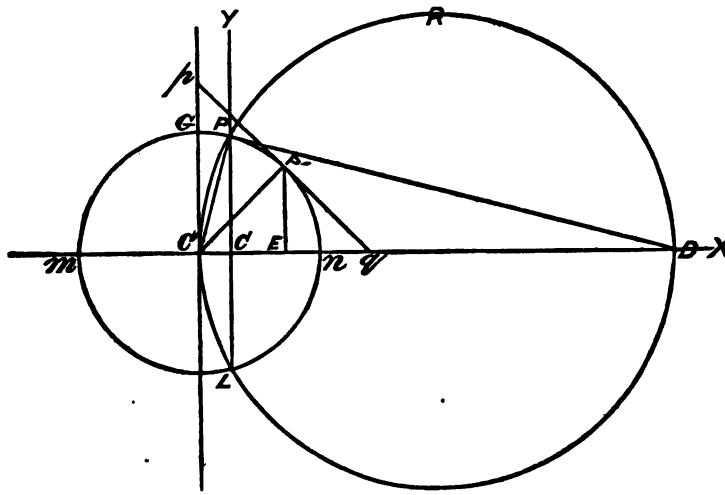


Fig. 8.

It was stated that a and e being given, it is possible to construct the preceding figure, aided by the results of demonstration. Since $e = \frac{ac}{b-c}$ it is best to start with the supposition that the distance between the forces and their relative intensities are given.

Let the line $C\,D$ represent the distance, and the relative intensities be such that $\frac{ac}{b-c} = C\,C'$. Then if the circle $C'\,R\,D$ be described with the diameter, $C'\,D$, it was shown to pass through the extremities of the minor axis of the curve of equal intensity, hence $P\,L$ passing through C is the minor axis. C' was shown as the centre of the major axis, and since the curve is a circle it may be described with $C'\,n$ or $C'\,P$ as a radius. $P\,D$ is the line which is tangent to the curve at the extremity of the minor axis and passes through D. $p\,q$ is the line tangent at 45° by which the resemblance to the circle was first noticed; $P'\,e$, the ordinate corresponding with the sine and $C'\,e$, or $x + e$ with the cosine. $P'\,C'$ is a normal line by which the curve was fully identified.

From the triangle $P\ C\ C'$ (Fig. 8) the expression $ae + e^2 = R^2$ may be derived, from which R, the radius of the circle of equal activity, may at once be obtained through a and e.

The similar triangles $P\ C\ C'$, $P\ C\ D$, and $P\ C'\ D$, may be used with profit to illustrate the agreement between algebraic and geometric relations.

The discussion thus far conducted pertains only to *one* plane upon the axis $M\ D$; but, since the same is true of *any* plane upon the given axis, it is true of all planes. If, therefore, the circle of equal intensity be caused to revolve about the axis $m\ n$, every point in the surface of the sphere thus generated is equally affected by the forces C and D.

The curve discussed in the preceeding propositions pertains to the *equality* of effects. Though not essential to the purpose in view, it appears proper to show by what modification a curve may be determined expressing a ratio of *unequal* effects,

$\frac{b}{(a-x)^2} = \frac{c}{x^2}$ is the expression of equal effect. If it be desired to find the curve where the effect from $b = n$ times the effect from c, the results derived from the solution of the equation $\frac{b}{(a-x)^2} = \frac{nc}{x^2}$, will furnish the data desired.

For equal effect $x = \frac{-a\,(c \pm \sqrt{bc})}{b-c}$

and $e = \frac{ac}{b-c}$.

For unequal effect $x = \frac{-\,a(nc \pm \sqrt{nbc})}{b-nc}$

and $e = \frac{anc}{b-nc}$.

From which it is evident that not only the values of x, but also e undergoes variation.

SECTION II.

Showing That the Curve Determined in Sec. 1 Correctly Represents the Orbit of a Body Moving About the Included Force as a Centre of Gravitation.

In every point of the circle, or the sphere of equal activity, the force at C acts with an intensity varying inversely as the square of the distance; for, that is the assumption upon which the discussion is based and by which the curve was developed,

$$\frac{c}{r^2} = \text{force of } C \text{ upon any point } P$$

$$\text{and } \frac{c}{r'^2} = \text{force of } C \text{ upon the point } P',$$

$$\text{hence } F \text{ or the force from } C, \propto \frac{1}{r^2}.$$

It is evident, then, that if any body describes the circle under the influence of C only, that force varies inversely as the square of the distance. And, though, the nature of the curve was determined in a discussion involving two forces, the *nature* of the force C is the same whether we consider it jointly with D or alone.

The analogy is so complete that C may at once be substituted for the sun and the circle of equal intensity becomes a planetary orbit, determined by the law of gravitation and the primary centrifugal impulse of the planet.

A detailed criticism on Newton's demonstrations is reserved for a later period, but with Fig. 9, as an illustration it will be seen how slight a mistake in the demonstration of a fundamental proposition has involved the entire doctrine of celestial mechanics in error.

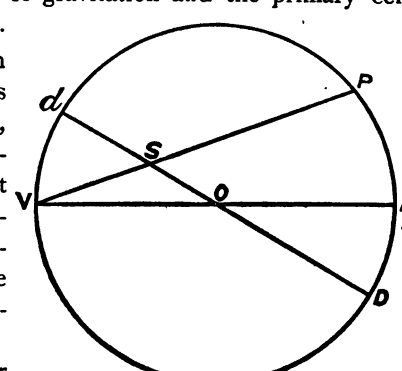


Fig. 9.

Newton is inquiring after the law of force tending from

any point of the circumference of the circle to a given point within. It is well understood that in cases of terms expressing variation, all constants should be omitted, and all variables retained. In the construction of his figure (of which Fig. 9 is a reproduction), Newton fixes upon the points P and S and orders the chord PV to be drawn. Now will any one say that a chord drawn through two fixed points is not a constant? At the conclusion of his demonstration, however, Newton declares that the force tending to $S \propto \frac{1}{SP^2\ PV^3}$ in which he retains the constant PV in the expression of variation. Had he stricken out this constant factor, his conclusion would be $S \propto \frac{1}{SP^2}$, conforming to the law of gravitation.

The case just stated illustrates the discussion through which we have just passed. For if $d\ D$ be regarded as the axis, $S\ O =$ eccentricity, $S\ P$ the radius vector. It was conclusively shown that the force upon the circumference varies inversely as the square of the distance. We should notice that $S\ P$ as a vector is variable; but as part of the chord $P\ V$ it is *constant.*

NOTE.—It has been objected that PV is not a *constant*, in answer to which see appendix.

SECTION III.

General Discussion Upon the Elements of Orbits and the Activities of Forces, Including the Conditions of Precipitation.

In its motion about the sun, the aphelion and perihelion distances of a planet will be modified (and determined in fact) by the relation of the force of gravity to the projectile power. It is possible therefore for the sun to occupy any position within the orbit from centre to circumference. In the treatment of orbits the forces will be distinguished by the usual terms *centrifugal* and *centripetal.*

The evidences furnished by the unvarying regulurity of planetary movement, as observed for centuries, give us reason to assume that the centrifugal force suffers no appreciable diminution; that it may be regarded as the perpetuation of an original impulse. It may some day also be pronounced a physical dependency upon some higher physical law. It does not interpose any special barrler, however, to accept its manifestations without certainly knowing its source. The *variations* of speed in the motions of planets are fully accounted for by the accelerating and retarding tendencies of the central force.

Suppose a planet to describe an orbit with the sun in the centre. Since the centripetal force varies inversely as the square of the distance; when the distance is constant the force is constant. The same power, then, that will cause the planet to follow any part of its orbit will cause it to follow the whole. There will be an equilibrium of forces at all times and the movement will be uniform during its entire course. Such orbits are called non-eccentric and will be used as the unit of comparison in the following demonstrations.

If the velocity of a member moving in a non-eccentric orbit be *diminished* it will describe an *ecceentric* orbit *inside* of the non-eccentric; if the velocity be *increased* it will describe an *eccentric outside* of the non-eccentric. The point at which the centrifugal momentum is increased beyond equilibrium will be the future *peri-*

helion. The point at which the centrifugal momentum is diminished when in equilibrium will be the future *aphelion.* Considered from a perihelion standpoint the eccentricity of an orbit is caused and measured by the *surplus* of centrifugal force as compared with a non-eccentric.* Considered from an aphelion standpoint the eccentricity of an orbit is caused and measured by the *deficiency* in centrifugal force as compared with the non-eccentric.† The perihelion will in these demonstrations be mainly observed.

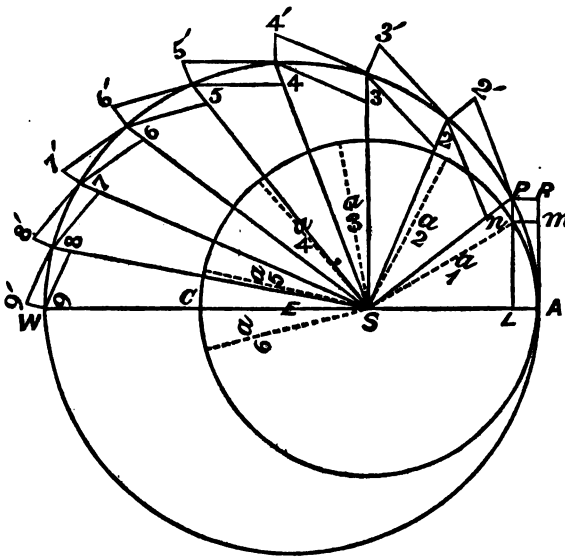


Fig. 10.

Fig. 10, embracing some results of future demonstration serves to introduce the discussions pertaining to the relations of *non-eccentric* to *eccentric* motion.

Suppose S to be the sun, A a planet at rest. Propel A with a force at right angles with SA and sufficient to carry A to m in the same time as the attraction of the sun would draw A to L; then, by mechanics, the *actual* position of A at the end of the given interval of time will be n. Sn being a radius of the same circle as SA, the centripetal force at n is the same as it was at A. Also, since the attractive force at A acts at right angles with the centri-

*A non-eccentric whose radius = perihelion.
†A non-eccentric whose radius = aphelion.

fugal impulse, the tendency is only to change the *direction* without altering the *quantity* of centrifugal motion; hence the same relation of forces that existed at *A* also exists at *n*, therefore *A* will describe uniformly equal *arcs* in equal times, and the radius vector equal *areas*. It is, of course, understood that in order that the explanation just given may satisfy *curvature*, the successive intervals of time must be conceived *indefinitely small*.

Suppose now, the force exerted upon *A* be greater than that just considered and sufficient to carry *A* to *R* in the same time as gravity draws to *L*, then will *A* at the end of the given interval of time be at *P*, a point entirely outside of the noneccentric orbit *A B C*, but upon the same ordinate produced and with the same abscissa.

When *A* is at *P*, the centrifugal tendency is in the direction of the straight line *A P*2′, while the centripetal tendency is in the direction *S P*. At this point the forces do not oppose at right angles and the tendency of the central force is both to change the *direction* and the *quantity* of centrifugal motion.

The retarding factor of centripetal force is the subject of future demonstration, the object at present being only to illustrate in accordance with demonstrable laws.

At the end of the second interval of time the planet will be at 2, having described the diagonal of the trapezium *P* 122′. It will be observed that the *diagonal* is less than the side *P* 2′, which represents the centrifugal tendency at *P*. This difference is explained by the retarding tendency of the central force, by reason of which each successive arc of the semi-orbit is less than that preceding.

The law of "equal areas in equal times" might be readily derived from Fig. 10, but that having been clearly established by others, and the law not being in dispute, it is useless to repeat, only it must be allowed, for it can also be clearly proven, (and I think it is not disputed) that there is no association between an ellipse and the law of equal areas. The *form* of an orbit is a matter determined independently of the law of equal areas. So there is nothing in the law of equal areas that stands in the way of supporting the doctrine of circular orbits. Indeed, we may conceive a body moving about a central force in a *square* orbit. Just suppose the central force instead of acting *constantly*, to operate by *jerks*; polygons will result, whose sides and angles depend on the inten-

sity of the *jerks* and the intervals between them. If the force were exercised at four equidistant points in the course of a revolution, the result would be a square orbit. In all these cases the law of equal areas in equal times would be verified the same as in *actual* orbits.

Pursuing the law of equal areas a little farther, suppose S a given force and A a given body. While A is moving in the straight line to B the force S is latent. At B sufficient force is exerted upon the moving body to cause it to move in the direction BC. The force from S necessary to produce such effect must be equal to the force with which the free body is moving. One-half of the momentum with which the free body is moving will be destroyed and the other half tends in the direction BC. Of the force exerted by S which is equal to the momentum of the free body at the time of interference, by regarding SB as the resultant of its tendencies in the directions BC and BE, the half tending to BE is destroyed in destroying one-half the momentum of the moving body, while the other half together with the half of original momentum still remaining to the moving, it will proceed in the direction BC with the same velocity as it moved from A to B. Repeating the same impulses at C, D and E the moving body will describe the sides of the square with uniform velocity. It is quite evident that if the velocity in the case just described is uniform, the radius vector also describes equal areas in equal times.

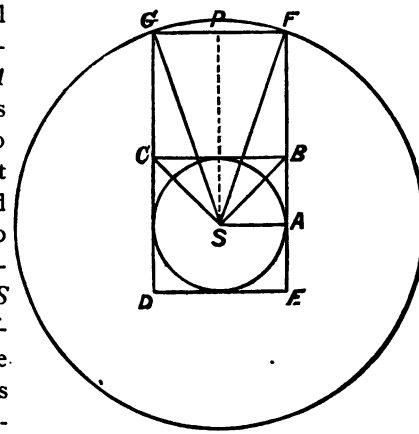


Fig. 11.

And if, from B the body be allowed to move uninterruptedly to F at which point its course is turned by reason of an impulse from S, so that its future course is in the line $F\,G$, then, if $B\,F = 2\,A\,B$ or $A\,F = 3\,A\,S$, in other words, if the angle $A\,F\,S = \frac{1}{3}\,S\,F\,G$,—then $F\,G$ represents $\frac{1}{3}$ of the force of the moving body at F. It will therefore move $\frac{1}{3}$ as fast in $F\,G$ as it did in $A\,F$ and, since its distance from S is three times as great at P as at A, the area

described by the radius vector is still equal in equal times. Once more, at G let an impulse from S change the direction to C, then by like reasoning the velocity will be again 3 times that of F to G, being the same as from A to F.

The conclusion to these observations upon the law of equal areas is that if a body moves uniformly in a straight line, the radius vector from *any point in space* will describe equal areas in equal times; but, if said motion be interfered with by any force whether of attraction or repulsion, the radius vector from the *disturbing center* only, will describe equal areas in equal times.

An examination of Fig. 10, reveals the following facts: If the velocity of the body at A by which it describes the noneccentric $A\ B\ C$, be increased, the resulting orbit is eccentric and larger, and the periodic time longer. It is seen also that in an eccentric orbit the tendency of the sun's attraction is resolved into a retarding or accelerating, and a deflecting factor.

THEOREM VII.

In any orbit the relation between centrifugal and centripetal force is constant.

For, the area described by the radius vector $\propto R^2$, and, since the area described by the radius vector is constant and equal to $\frac{1}{2} R$ multiplied by the included arc, in order to sustain the equality of areas the *arc* must $\propto \frac{1}{R^2}$. This result is a necessity from the law of equal areas. We have now derived that the *arc described in a unit of time* $\propto \frac{1}{R^2}$; but, *the force of gravitation also* $\propto \frac{1}{R^2}$ hence, the arc described in a unit of time $\propto$ as the force of gravitation; hence their ratio is constant, and since the arc described in a unit of time is the measure of centrifugal force, and gravitation represents centripetal force, the proposition is proven.

Theorem VII may also be proven and illustrated as follows.

Area described by the vector $\propto R^2$.

Area described by the vector $\propto$ arc.

But, the area is constant for all values of R, hence arc $\propto \frac{1}{R^2}$, that

NOTE.—In Prop. VII R represents radius vector.

is, if the radius vector is twice as long, the arc must be ¼ as great to satisfy the law of equal areas. Gravity $\propto \frac{1}{R^2}$ and the arc $\propto \frac{1}{R^2}$, hence if there is twice a former radius there is ¼ of the former centripetal force and ¼ the centrifugal force; ¼ of the former arc to deflect and ¼ of the former deflecting force. Prop. VII accounts for the constancy of curvature in any orbit however eccentric.

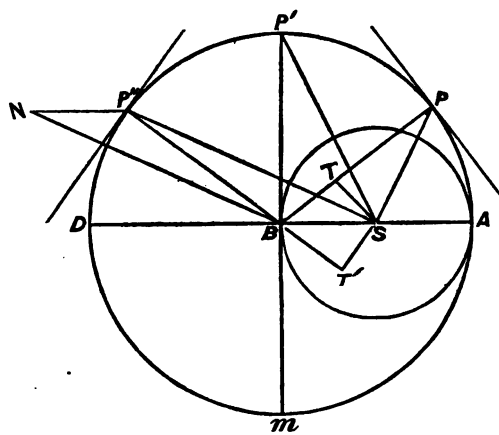


Fig. 12.

A PROBLEM.

A body is moving in the eccentric orbit $A\ PD$ about a force S; required to find the *retarding tendency* of the centripetal force.

As constructed, $B\ S$ represents the eccentricity of the orbit of which B is the center. When the body is at P centripetal force tends in the *direction* $P\ S$. That part of force which tends in the direction of the center B serves to deflect the path of the body from the tangent and that which tends in the direction of the tangent serves to accelerate when it draws in the direction that the body is moving and to retard when it draws in the opposite direction. After $B\ P$ draw $S\ T$ parallel to the tangent at P. If, now, $S\ P$ represents the *quantity* of centripetal force it may be regarded as the resultant of $P\ T$ the deflecting and $S\ T$ the retarding factor. $S\ T$ being parallel to the tangent at P is perpendicular to $B\ P$, $\therefore S\ T^2 = S\ P^2 - P\ T^2$. Most generally expressed the *retarding*

tendency of centripetal force *varies* as the *eccentricity* of the orbit and also as the *arc of revolution* measured from *perihelion*. Considered with reference to a *particular* orbit, the eccentricity being a constant, the retarding factor $\propto$ the arc of revolution. When the arc of revolution is o, $S\,P$ and $P\,T$ coincide and are equal, which changes the equation $S\,T^2 = S\,P^2 - P\,T^2$, to $S\,T^2 = o$; that is, at A there is no retarding tendency, which conclusion is both evident and agreeable.

When the body is in quadrature at P', $P'\,S$ the centripetal force is the component of $P'B$ deflection and $B\,S$ retarding. But $B\,S$ is the eccentricity of the orbit and $B\,P'$ its radius, hence at quadrature we derive the proportion Ret. force : Def. force :: Eccentricity : Radius of orbit.

$\frac{\text{Def. force}}{\text{Ret. force}} = \frac{\text{Radius}}{\text{Eccentricity}}$: — the ratio to which the retarding force has gradually attained from 0 at A. The relative value of the retarding factor attains its maximum near P' from which it again declines to 0 at D. The force at P'' may be resolved into $P''\,B$ deflective and $S\,B$ partly retarding and partly deflective. $S\,B$ by farther resolution $= S\,T'$ parallel to the tangent at P'', strictly retarding and $B\,T'$ deflective. The entire deflecting tendency from S at P'' is $P''\,B + B\,T'$ while the retarding tendency is $S\,T'$. These terms show a ratio of greater inequality between the deflecting and retarding forces at P'' than at P'.

The value of the retarding force depends on the angle formed at any part of the orbit by the radius and the radius vector of the orbit, from which also it might have been shown that the retarding force attains its maximum near P'. From D through M to A this interfering force acting *in the direction* of the body's motion tends to *accelerate* by reason of which the body returns to A with the same velocity after each revolution.

As a result of the solution of this problem of force it is evident that the greater the eccentricity of an orbit, the greater the variations in the velocity of the body pursuing it. Also, no matter how great the eccentricity, the body will not be deprived of all its centrifugal momentum before reaching aphelion. The following conclusion is also warranted:

That when the eccentricity of an orbit is very great, the body will pass its perihelion with enormous velocity, and by reason of the retarding factor in the central force, will spend much of its time in the remote parts of its course.

The early part of these discussions proves that if a body moves in a curve drawn toward a point within by a force which varies inversely as the square of the distance, that curve is a circle. The case will now be examined inversely, as follows:

THEOREM VIII.

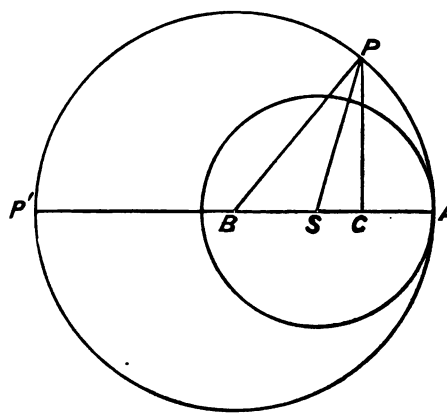


Fig. 13.

If a body moves in a circular orbit about a force within, that force varies inversely as the square of the distance.

Let $A\,P\,P'$ be the given orbit, S the position of the force within, B the centre of the orbit and P the position of the moving body at any time; $B\,S$ represents the eccentricity of the orbit, $B\,P$ its radius and $S\,P$ its radius vector to P. Suppose the body to move from A to P in a unit of time, then $C\,A$ is the *measure of attraction*, $C\,P$ being perpendicular to $A\,P'$.

From properties of the circle we have

$$P'C : P\,C :: P\,C : AC,$$

or $2R - C\,A : CP :: CP : A\,C$. Then by trigonometry,

$$2R - \text{ver sin} : \sin P\,B\,A :: \sin P\,B\,A : \text{ver sin}.$$

NOTE.—See Appendix, Art. II.

When the angle PBA is infinitely small, ver sin may be omitted from the first term, then $2R$: sin :: sin : ver sin.

$$2\text{ver sin} = \frac{\sin^2}{R}.$$

From principles in mechanics, the effect of attraction (in this case ver sin) $= \frac{1}{2}$ force t^2, but t or time in this case is unity, hence

$$\text{Force} = 2 \text{ ver sin} = \frac{\sin^2}{R}.$$

Let r represent the radius vector and H the area described in a unit of time. When the arc is infinitely small, sin = arc, hence $H = \frac{1}{2}r \times \text{arc} = \frac{1}{2}r\sin$; whence $\sin = \frac{2H}{r}$ $\therefore \sin^2 = \frac{4H^2}{r^2}$.

Substituting this value of $\sin^2$ in the expression, force $= \frac{\sin^2}{R}$, we derive, force $= \frac{4H^2}{Rr^2}$. Since H and R are *constants*, force $\propto \frac{1}{r^2}$ which is the proposition.

THEOREM IX.

When two or more bodies move in circular orbits of equal eccentricity about the same force, varying inversely as the square of the distance, the squares of the periodic times vary as the cubes of the radii.

Let T denote the periodic time of a planet in days and H the area described by the vector in one day, then the entire area of the orbit $= TH$. But the area of the orbit is also πR^2, hence $TH = \pi R^2$, or $H = \frac{\pi R^2}{T}$.

By the preceding proposition the force, or F, $= \frac{4H^2}{Rr^2}$. By substituting the value of H just found, $F = \frac{4\pi^2 R^4}{T^2 Rr^2} = \frac{4\pi^2 R^3}{T^2 r^2}$.

Let f = the intensity of the centripetal force at the distance of unity, then

$$f : F :: \frac{1}{1^2} : \frac{1}{r^2};\ F = \frac{f}{r^2}.$$

But, $F = \frac{4\pi^2 R^3}{T^2 r^2}$ $\therefore \frac{f}{r^2} = \frac{4\pi^2 R^3}{T^2 r^2}$, whence

$$T^2 = \frac{4\pi^2 R^3}{f} \text{ and } T = \frac{2\pi R^{\frac{3}{2}}}{\sqrt{f}}.$$

If another planet revolves around the same force in a similar orbit, whose radius is R', then will $T' = \frac{2\pi R'^{\frac{3}{2}}}{\sqrt{f}}$, from which

$$T : T :: R^{\frac{3}{2}} : R'^{\frac{3}{2}},$$

or $T^2 : T'^2 :: R^3 : R'^3$, which is the proposition.

Theorems VIII and IX are among those which conflict with Newton's demonstrations. Both of us cannot be correct. Criticism later.

THEOREM X.

If a body describes a non-eccentric orbit about the sun the centrifugal and centripetal forces are equal.

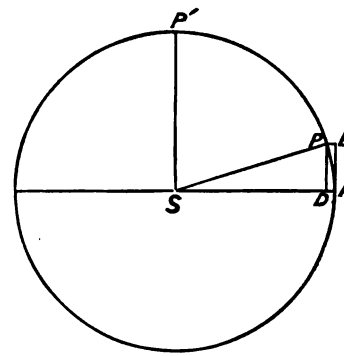


Fig. 14.

Let PA be the arc described in a unit of time, then AB or the sin PD represents the centrifugal force and the ver sin. DA represents centripetal force. As the arc AP is diminished, sin. and ver sin. approach equality to which they fully attain when the arc is 0, hence the forces which they represent are equal at A, which is the proposition.

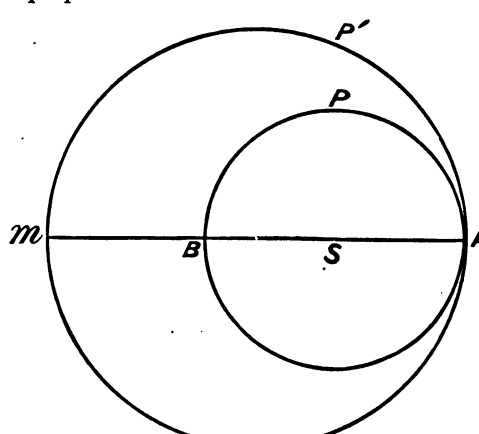


Fig. 15.

If APB represents a non-eccentric orbit about the central force

S, it is clearly shown by what precedes that if the centrifugal force of the moving body be increased at A, it will describe a *larger* orbit about the central force. The problem now before us is to determine the changes in the elements of an orbit corresponding to a given change of velocity or distance.

Let us examine the result derived by Theorem IX.

$t^2 : t'^2 :: R^3 : R'^3$, which is, as will be shown, true of all orbits, whether eccentric or non-eccentric; that is, if S be the position of the sun, and No. 1, 2 and 3 are the orbits of planets, the

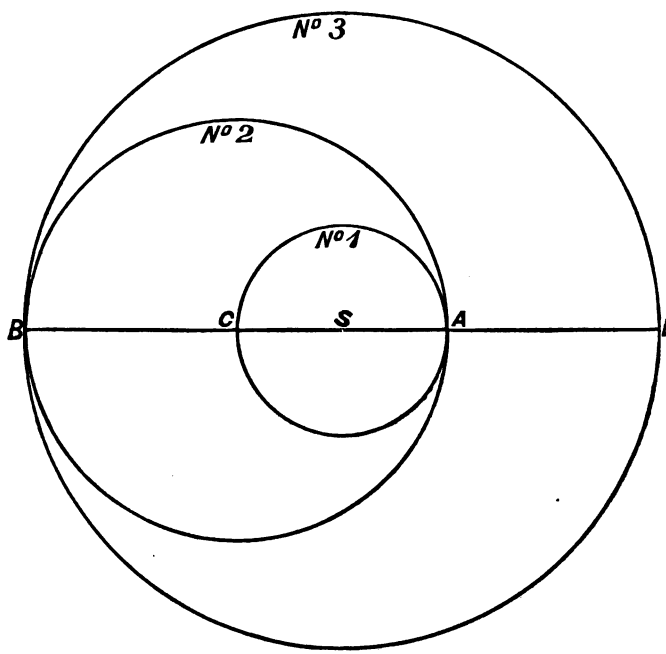


Fig. 16.

foregoing proposition applies in the comparison of Nos. 1 and 2 of which No. 1 is non-eccentric and No. 2 eccentric, as well as to a comparison of Nos. 1 and 3, which are concentric.

Suppose No. 1 to represent unity in all its elements, and that the radii of 2 and 3 are respectively 2 and 3 times that of No. 1. If in the proportion $t^2 : t'^2 :: R^3 : R'^3$, t and R represent the time

and radius of No. 1, and t' and R', the time and radius of No. 2, then, since t and R are unity,

$$1^2 : (2.83 -)^2 :: (1)^3 : (2)^3.$$

The area of No. 2 is 4 times that of No. 1, since area $\propto R^2$.

The area of No. 2 being 4 times and the periodic time 2.83 — times that of No. 1, the area described by the radius vector of No. 2 = 4 ÷ 2.83 = 1.41 ; that described in No. 1 being unity.

At A the planets describing Nos. 1 and 2 move by a common vector SA, hence, if the area described by the vector of No. 2 = 1.41 times that described in No. 1, it follows, that, at A, the velocity of the planet describing No. 2, = 1.41 times the velocity of that describing No. 1.

At B, however, the body describing No. 2 moving on a longer vector, must, in order to satisfy the law of equal areas, move slower. The vector SB being 3 times SA the velocity in No. 2 at B must be $\frac{1}{3}$ as great as at A, or $\frac{1}{3}$ of 1.41 = .47 of the uniform velocity in No. 1.

From this explanation we are warranted in making the following declaration :

If No. 1 represents the orbit of a planet about the sun, an increase of .41 of its velocity will result in an orbit of twice the former radius, four times the former area, 2.82 times the former time. The point at which the velocity was increased, is the future perihelion of the orbit. The eccentricity of the orbit is half the difference between perihelion and aphelion, which in the case of No. 2 equals the radius of No. 1.

Comparing Nos. 1 and 3 by the proportion $t^2 : t'^2 :: R^3 : R'^3$, we have,

$$(1)^2 : (5.19 +)^2 :: (1)^3 : (3)^3.$$

Three times radius of No. 1 = 9 times area, which, being described in 5.19 times the time, gives the area described in a unit of time = 9 ÷ 5.19 + = 1.73 ; that is, the area described in a unit of time by the radius vector of No. 3 = 1.73 times that described by the vector of No. 1.

But, since the radius of No. 3 = 3 times that of No. 1, the velocity in No. 3 = $\frac{1}{3}$ of 1.73 = .58 — times the velocity in No. 1.

Nos. 1 and 3 being concentric orbits, the relations above expressed apply to them at all points.

Comparing Nos. 2 and 3 in the proportion $t^2 : t'^2 :: R^3 :: R'^3$ it becomes

$$(2.83-)^2 : (5.19+)^2 :: (2)^3 : (3)^3 \text{ that is}$$

$$\frac{5.19}{2.83} = 1.83 \text{ is the ratio of the } \textit{times} \text{ of Nos. 2 and 3, or}$$

the time in No. 3 = 1.83 times the time of No. 2. But the period of No. 3 is 5.19, hence that of No. 2 = 5.19 ÷ 1.83 = 2.82, the same as found by a comparison of Nos. 1 and 2. More *properly* and in a *general* way, when three terms of the proportion $t^2 : t'^2 :: R^3 : R'^3$ are given, the fourth is found by the rules of proportion.

Since the time in No. 3 = 1.83 times that in No. 2, and the area of No. 3 = $\frac{9}{4}$ times that of No. 2, the area described in a unit of time in No. 3 is $\frac{9}{4} \div 1.83 = 1.23$ times that in No. 2. At B they move with the common vector BS, therefore the *velocity* in No. 3 at B = 1.23 times that in No. 2. But the velocity in No. 2 at B was found = .47 by which that if No. 3 at B = .47 × 1.23 = .58—, which is the same as aphelion velocity of No. 2 as compared with No. 1.

These comparisons warrant also the following conclusions: If No. 3 represents the orbit of a planet about the sun (non-eccentric), by reducing the velocity of the body from .58 to .47, or about 20 per cent., the resulting orbit is an eccentric whose area = $\frac{4}{9}$, period $\frac{1}{1.83}$, and eccentricity equal to the difference of radii in this case 1. The point at which the velocity was diminished will be the future aphelion of the resulting orbit.

The proposition $t^2 : t'^2 :: R^3 : R'^3$, when t and R are regarded as unity, may be put in the form $1 : t' :: 1 : R'^{\frac{3}{2}}$.

Whence, periodic time, $t \propto R^{\frac{3}{2}}$ (1).

Area of an orbit, $a \propto R^2$ (2).

Area in a unit of time, $a' \propto \frac{R^2}{R^{\frac{3}{2}}} = \sqrt{R}$ (3).

Non-eccentric vel. $V \propto \frac{1}{\sqrt{R}}$. (4).

It need scarcely be mentioned that if the radius of an orbit be known, by the use of these formulæ all the elements except eccentricity may be directly determined. In cases where periodic time is better known than radius, the latter may be found by the first

formula. When neither t nor R are known, perihelion distance and velocity will furnish the area described in a unit of time from which by formula (3) R may be found and then in turn the other elements.

Other expressions of variation may be derived from those given, and it seems proper to give in this connection the law of variation between centrifugal and centripetal force. At 4 times a given distance from the sun the hourly velocity is $\frac{1}{2}$ as great; at 9 times the distance $\frac{1}{3}$ as great, and generally, at n times the distance $\frac{1}{\sqrt{n}}$ times as great. Gravitation at 4 times distance $= \frac{1}{16}$; at 9 times, $\frac{1}{81}$; at n times $\frac{1}{n^2}$. But since gravitation is the same as centripetal force and velocity is the measure of centrifugal force, we obtain the expressions, centrifugal force $\propto \frac{1}{\sqrt{n}}$;

$$\text{Centripetal force} \quad \propto \frac{1}{n^2};$$

$$\text{Centripetal force} \propto \overline{\text{centrifugal}}^4.$$

As an illustration of this variation it may be stated that if in a certain orbit the forces are in a relation of equilibrium, when the distance is increased to 4 times that in the given orbit, centripetal force will be $\frac{1}{16}$ and centrifugal $\frac{1}{2}$ as great as before, in which relation they will be again in equilibrium. The importance of the formulæ just derived is not so evident in planetary orbits, but their special importance will be recognized in the treatment of cometary orbits.

Considering the proportion $t^2 : t'^2 :: R^3 : R'^3$ in its most general sense it is impossible to derive a formula for the eccentricity of an orbit, for, it matters not whether orbits No. 2 and 3 of Fig. 16 be concentric with No. 1, in which case there is no eccentricity, or whether they be eccentric with a common perihelion or different perihelia. A limiting qualification of the general proportion is therefore necessary. In all eccentric orbits the basis of comparison is the non-eccentric orbit drawn upon the point of perihelion in the eccentric. Two or more orbits are said to be equally eccentric

when their eccentricity represents the same part of their own perihelion.

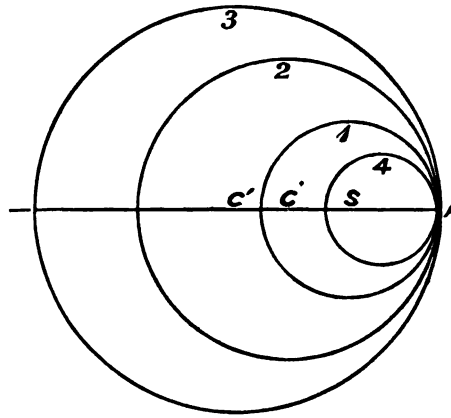


Fig. 17.

From what has been demonstrated it is clear that if S Fig. 17 is the position of the sun, and A the point at which a body receives its centrifugal impulse, if that impulse be such as to establish an equilibrium between centripetal and centrifugal force, the body will describe the non-eccentric orbit No. 1. By a greater impulse at A the result is No. 2, and by still greater No. 3. When centrifugal force is less than that of No. 1, the result is a negatively eccentric orbit within, as No. 4. As a law the proportion $t^2 : t'^2 :: R^3 : R'^3$ applies to all these orbits, of which No. 1 is the most convenient basis of comparison.

It is needless to repeat the operations performed in illustration of Fig. 16 as the value of the *eccentricity* may at once be taken from the relations expressed in Fig. 17. If SA represents the radius of No. 1, and CA the radius of No. 2, computed from known elements, then $CA - SA = SC$ is the eccentricity of No. 2. In like manner $C'S$ is the eccentricity of No. 3. But, when Nos. 1 and 2 are compared in the proportion $t^2 : t'^2 :: R^3 : R'^3$, R represents the radius of No. 1, and R' that of No. 2, hence, the eccentricity of an orbit is derived at once from the proportion when limited to a common perihelion, and equal to $R' - R$. When R is greater than R' this value is negative, as in the case of No. 4.

It has been shown that in No. 1, centrifugal and centripetal force are in a relation of equilibrium, hence the velocity of No. 2 at

A—velocity of No. 1 may be considered an excess or surplus of centrifugal force at that point, and, since this difference determines the eccentricity of No. 2, or any other orbit of the same perihelion, we may define the eccentricity of an orbit as *the measure of surplus centrifugal force at perihelion.*

It must be remembered that in an eccentric orbit centrifugal force does not continue in excess. The retarding tendency of centripetal force drains upon this surplus, so that at aphelion the order is reversed, the velocity of the eccentric being less than that of the non-eccentric of that distance, as was shown in Fig. 16.

In the geometry of the ellipse, the eccentricity is defined as the distance from the centre to the focus divided by the semi-major axis. The eccentricities of orbits have been computed in the same way. For the sake of harmony with the other elements which are derived by comparison with the non-eccentric, we will regard the eccentricity of our orbit as a part of *perihelion distance*, which is also the radius of our standard of comparison. When we say, then, that the eccentricity of an orbit is $\frac{1}{4}$, we do not mean $\frac{1}{4}$ of the radius of said orbit, but $\frac{1}{4}$ of its perihelion. It would be possible to adapt to the former method; but, believing that the harmony and simplicity gained will more than compensate, we make the change, which is quite easy.

If c represents the focal distance from the centre, and A the radius of an orbit $\frac{c}{A}$ is the value of the eccentricity as generally treated, but in the calculations pertaining to the circular theory $\frac{c}{A-c}$ or the focal distance divided by perihelion will be regarded as the value of eccentricity. As an example, take the earth's orbit whose eccentricity is .01677, by the formula of transformation it becomes $\frac{.01677}{1-.01677} = .01627$. By the same method the entire table of planetary eccentricities is easily adjusted to agree with the circular method.

By means of the various formulæ expressing the relations of the elements, tables may be constructed from which at a glance may be seen the result of a given change from equilibrium.

THEOREM XI.

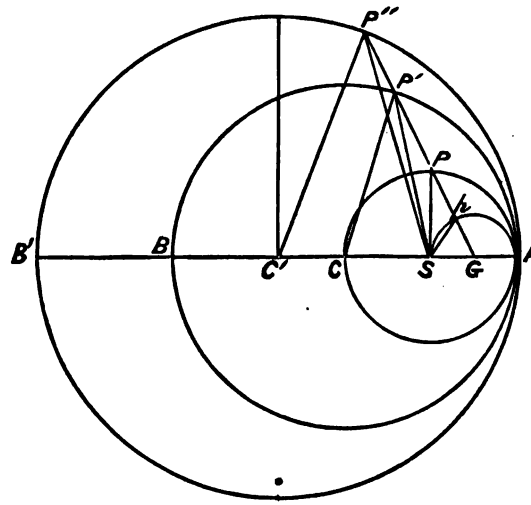


Fig. 18.

The mean velocity in an eccentric orbit is indicated by the apex of an isosceles triangle whose base is the eccentricity of the orbit, and its sides the radius and vector of the orbit.

If S be the position of the sun and $A\,P$ a non-eccentric and $A\,P'$ an eccentric orbit, then will the velocity in the orbit $A\,P'$ have a mean value at P'. Since the velocity varies inversely as the radius vector, when the vector has a mean value velocity is also a mean.

In the orbit $A\,P'$, $C\,S$ represents eccentricity and is the base of the isosceles triangle whose sides are the radius $C\,P'$ and the radius vector $S\,P'$. $S\,P'$ being the numerical mean between $S\,A$ and $S\,B$ the velocity at P' is the mean velocity of the orbit $A\,P'\,B$. The *mean* velocity of an orbit might be defined as the uniform velocity belonging to a non-eccentric of the same size.

If orbits of greater or less eccentricity be taken it will be observed that the point at which they cross the line drawn from the middle of $S\,A$ through P is the point of mean velocity, $P\,S$ being perpendicular to $A\,B$.

Since the *limit* of eccentricity is the *radius* of an orbit, the point of mean velocity can not be less but must be greater than 60°, and

since negative eccentricity can not be greater than $\frac{1}{2}$ radius, the isosceles triangle $Sp\,G$ is equiangular, hence, the arc $Ap = 120°$. This places the limits of mean motion between the 60° and 120° of an orbit, depending upon the degree of its eccentricity.

THEOREM XII.

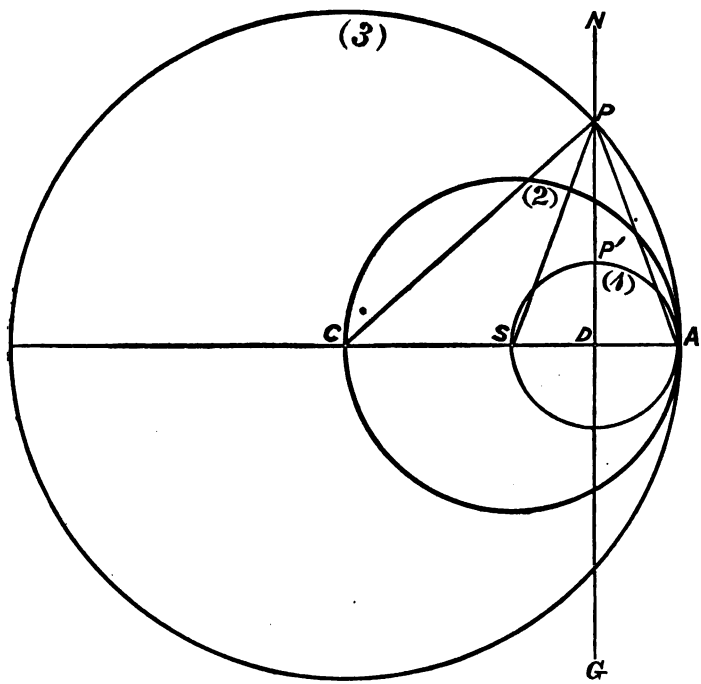


Fig. 19.

If a line be drawn perpendicular from the middle of the perihelion distance of an orbit, the velocities of all the eccentric orbits of that perihelion will be equal to the non-eccentric velocity at the points where they cross said line.

If S be the position of the sun and SA the given perihelion distance, of which D is the middle, No. 2 being the non-eccentric orbit, No. 3 an eccentric and No. 1 the limit of negative eccentricity, then will the velocities in all these orbits be equal to the non-eccentric at the point where they cross the line DN.

Suppose No. 3 an orbit whose radius is four times that of No. 2, then, as has been found, the perihelion velocity in No. 3 = 2 times that of the non-eccentric No. 2. By the law of equal areas the velocity in No. 3 will be equal to that of No. 2, where its radius vector is twice SA. Suppose P to be that point, then let $SA = 1$, $SC = 3$, $SP = 2$, and $CP = 4$. Draw PA and assume SPA an isosceles triangle, then is PD its altitude, $PD^2 = PS^2 - SD^2 = \frac{15}{4}$; $CD^2 = \frac{49}{4}$; $CD^2 + PD^2 = CP^2 = \frac{49}{4} + \frac{15}{4} = 16$, whence $CP = 4$, which is its known value; hence the supposition is true that SPA is an isosceles triangle. Knowing the eccentricity or periodic time of an orbit, the point P may be determined without constructing the orbit. Applying the same method to other given orbits of the same perihelion, it will be found that the point in question invariably falls upon the given perpendicular.

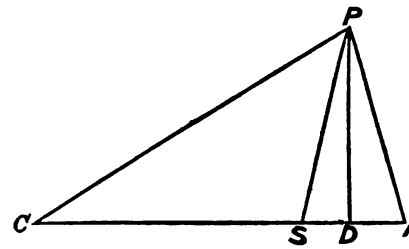


Fig. 20.

In the limiting orbit No. 1 it will be observed that the perpendicular bisects its diameter, whence it cuts the orbit at 90° from A. As the orbit increases in size the value of the arc from A to the point of intersection with the perpendicular diminishes. In very eccentric orbits this point is reached in comparatively few degrees. By diminishing perihelion and increasing eccentricity, it will be observed that the minimum limit of this arc is zero. The limit of maximum arc is the orbit of precipitation (No. 1), and the minimum limit the infinite circle or motion from A perpendiular to SA.

The points determined by Theorems XI. and XII., though of little value in relation to planetary orbits, will help to develop points of interest in the investigation of cometary orbits.

THEOREM XIII.

If a body describing a non-eccentric orbit about the sun were deprived of about $\frac{1}{4}$ of its velocity the consequencc would be immediate semi-circular percepitation into the sun.

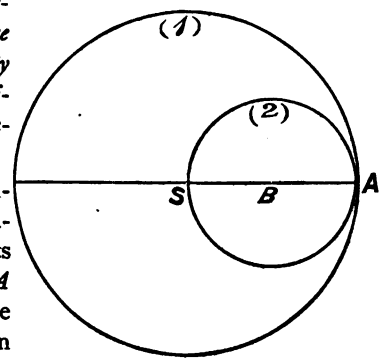


Fig. 21.

Suppose S, Fig. 21. the position of the sun; No. 1 a non-eccentric orbit whose elements are unity. If at any point as A the velocity of the body be diminished it will describe an orbit of negative eccentricity within No. 1. Finally the orbit tends through the centre of the sun, which condition will be termed the limit of precipitation. Taking into account the magnitude of the sun it is evident that precipitation will result when the orbit tend within radius or 444,000 miles of the centre.

The proportion $t^2 : t'^2 :: R^3 : R'^3$ does not exclude any orbit, however eccentric, and applies to Nos. 1 and 2 of Fig. 21 as truly as to any other case. The elements of No. 1 being unity the proportion becomes $1^2 : (\frac{1}{2.88})^2 :: 1^3 : (\frac{1}{2})^3$.

The radius of No. 2 being $\frac{1}{2}$ of No. 1 its area is $\frac{1}{4}$ of No. 1. Since the periodic time of No. 2, in case the sun did not arrest, is $\frac{1}{2.88}$, the area described by the vector in No. $2 = \frac{1}{4} \div \frac{1}{2.88} = .70+$. At A the areas in Nos. 1 and 2 being described by the same vector SA their velocities are as the areas described; that is, the area described in No. $2 = .70$ of that of No. 1.

Thus we see that the body describing No. 1 were deprived of $\frac{3}{10}$ of its centrifugal force its path would tend through the centre of the sun. Taking into account the radius of the sun, precipitation will take place at a loss of about $\frac{1}{4}$ of any planets centrifugal force.

THEOREM XIV.

The velocities of precipitation from the planets to the sun vary as the square root of their distances.

An application of the rule of precipitation derived from Theorem XIII. furnishes the following table of approximate values, giving the velocities with which in case of precipitation the several planets would approach the sun when yet 1,000,000 miles distant from the centre:

Mercury,	at 1,000,000	miles,	approaches	at the	rate of	2,800,000.
Venus,	" 1,000,000	"	"	"	"	3,800,000.
Earth,	" 1,000,000	"	"	"	"	4,500,000.
Mars,	" 1,000,000	"	"	"	"	5,500,000.
Jupiter,	" 1,000,000	"	"	"	"	10,000,000.
Saturn,	" 1,000,000	"	"	"	"	14,000,000.
Uranus,	" 1,000,000	"	"	"	"	20,000,000.
Neptune,	" 1,000,000	"	"	"	"	25,000,000.

A comparison of these values with the distances of the several planets, discloses that the rates of precipitation vary as the square root of the distance; thus, Saturn, whose distance is about nine times that of the earth, precipitates with a velocity about three times that of the earth. The same test being verified in every available case, it may with safety be announced as a law, that in case of precipitations of the several planets into the sun, their velocities vary as the square root of the distance.

But this law may be derived also directly from the formulæ obtained from $t^2 : t'^2 :: R^3 : R'^3$.

For $V \propto \frac{1}{\sqrt{R}}$ (Formula 4, Fig. 16), which is also true of the starting velocities in the orbit of precipitation. The acquired velocity at the time of precipitation varies as the distance from which the body started. We have, therefore, a double variation. In the first place the velocity of precipitation $\propto \frac{1}{\sqrt{R}}$ and in the second $\propto R$, which variations being compounded, results in the expression that the velocity of precipitation $\propto \frac{R}{\sqrt{R}} = \sqrt{R}$ which is the proposition.

A body to revolve about the sun at a distance of 500,000 miles must move about 1,000,000 miles per hour, its velocity of precipitation being about 700,000. Not being far from the surface of the sun, it would strike the sun with a velocity something like 750,000 miles per hour. In the case of precipitation from so near a quarter it must be remembered that since the body strikes the sun obliquely the force must undergo resolution into centrifugal and centripetal tendency, that part which tends in the direction of the sun's centre accounts for gravitation, while that which tends in the direction of a tangent is the relic of centrifugal force at the time of contact. In the case of approaches from great distances the direction of contact is almost perpendicular to the surface of the sun, hence the force may be altogether accounted as the accumulated result of gravitation.

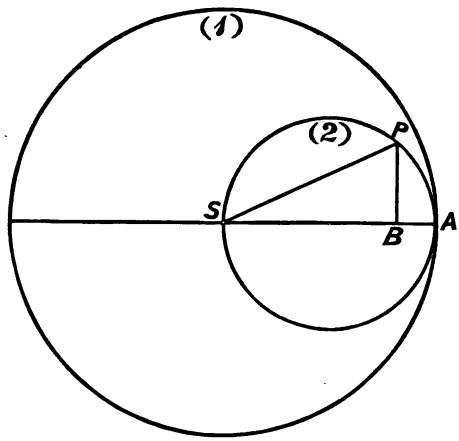


Fig. 22.

The time in which precipitation in the limit of orbital motion takes place is determined by the proportion $t^2 : t'^2 :: R^3 : R'^3$ in which for any particular planet t and R being unity, R' is $\frac{1}{2}$ and $t' \frac{1}{2.83}$ which is twice the time for precipitation, hence, the periodic time of any planet divided by 5.66 is the time of precipitation by the semi-circle No. 2. In the case of the earth the period would be about 64 days.

It will also appear that the period of precipitation by the semi-circle $A\ P\ S$ is considerably longer than that of direct approach in

the line AS. Within narrow limits where gravitation may be regarded as a *constant* force, horizontal motion does not altar the time of precipitation, but at great distances where the variation of gravitation is considered the case is different. Suppose in Fig. 22, that by gravity a body would fall from A to B in a unit of time, then at the next interval the force of gravity would be increased by the difference between SA^2 and SB^2, and would act with an intensity represented by $\frac{1}{SB^2}$. In the case of circular motion however, the body at the end of the first interval of time will be at P, so that the force of gravity is increased only by the difference between SA^2 and SP^2. Intensity for the second interval of time is represented by $\frac{1}{SP^2}$ which is considerably less than $\frac{1}{SB^2}$.

The demonstrations pertaining to orblts might properly rest with precipitation, but an inspection of Fig. 19 suggests a few more observations which are to be judged by their own merits.

It has been shown that if the velocity of the revolving body at A be reduced by $\frac{3}{10}$ it will follow the semi-circle of precipitation APS. Should the body at A be deprived of all its centrifugal motion it would evidently precipitate in the direction ADS. Any degree of velocity from 0 to .70 would furnish a curve of precipitation between the right line ADS and the semi-circle $AP'S$. The curve is an arc of a circle whose centre is somewhere in the line DG, its circumference invariably passing through S and A. As the velocity decreases the centre of the circle of precipitation recedes from D. SA is then a chord subtending the arc of precipitation. As centrifugal force approaches zero, the radius to the arc of precipitation approaches infinity, at which limit the arc and chord are indentical, and equal to SA.

We have found Theorem XIV that the velocity of precipitation $\propto \sqrt{R}$. This expression being independent of centrifugal motion it is presumed that the velocity of precipitation is the same, whether by the curve or the straight line. As a matter of *time* it is evident that precipitation from a state of rest will occur in the shortest period. By reference to Fig. 19 it will also be observed that when bodies fall from a great distance into the sun the velocity acquired at midway is equal to the non-eccentric velocity belonging to the

starting point. For example, if the earth should from a state of rest drop into the sun, the velocity acquired at midway would be 68,000 miles per hour, which is the planet's hourly motion in its orbit. It was also shown that the time of semi-circular precipitation varies as $R^{\frac{3}{2}}$. The time of precipitation from rest is not the same as the time by semi-circle, but evidently the times vary directly; that is, the time of falling from great distances into the sun varies as $R^{\frac{3}{2}}$. It might just here be remarked that the formulæ derived by Newton and others are based upon the notion that the law of gravitation demands elliptical movement. Their merits therefore rest upon a common basis. The relation of the time of precipitation by the semi-circle to the time from rest is worthy of discovery, but not essential to the present purpose.

In the event of precipitation there are various suppositions to be made concerning the behavior of bodies passing through the centre of force. The requirement in these suppositions is that, while the influence of gravitation should emanate from S, Fig. 23, the precipitating bodies should not be arrested but be allowed to pursue their course subject to the same central force.

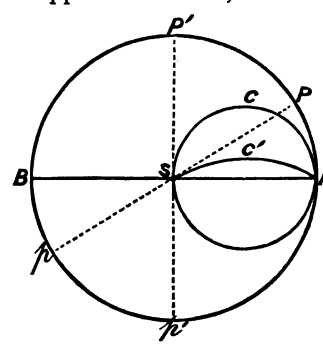


Fig. 23.

The first case is a body at A falling from a state of rest. It is not in contradiction with accepted theory that, under these circumstances, the body will arrive at S with momentum sufficient to carry it to B, at which point it will be brought to rest, after which it will return through S to A. The vibrations, if unimpeded, will occur with the same permanency and regularity as pertains to orbital movement. This case is one of conformity with the simple principle that if a ball be cast into the air perpendicularly, the momentum ascending is the same as the momentum descending. It is illustrated by supposing the body to be cast from S with sufficient force to reach A, whence it will return to S with its original impulse. By reason of the tendency to move in a straight line, the body moves to B, a distance equal to $A\,S$, and so on indefinitely.

The other cases to be considered are in conflict with authority.

The second case to be observed is the limit of precipitation, in which the body describes a semi-circle on the diameter $A\,S$. In all orbital motion, centripetal force tends to accelerate in the direction of the radius vector, while centrifugal tends in the direction of a tangent at the extremity of the vector. So long as these forces do not tend in the same direction, the path of the moving body remains circular. In very eccentric orbits these forces tend more and more nearly in the same direction as eccentricity increases, and in the case of precipitation they tend in the same direction exactly as the body passes through the centre of force. After the body has passed the central force the tendency is no longer to follow a curve but a straight line. Since the semi-circle meets $A\,B$ at right angles, the future tendency is in the straight line $S\,p'$. In conformity with habits laid down in case 1, this body will now vibrate in the line $p'\,P'$.

Since these cases are the extreme conditions of precipitation, the result of all other degrees of velocity will be precipitation at angles with $A\,S$ varying between zero and 90°, and final vibrations between the first and third quadrants as $p\,P$. Since the body moving from A precipitates with the same velocity, whether from rest or motion, the final vibrations will be confined within the limits of the non-eccentric orbit $A\,P\,B$. According to Newton's theory, in all these cases, except the first, the body from A will describe an ellipse, with S in the remote focus, the ellipse being narrower as the centrifugal force becomes weaker. Critical examination of this case is deferred for later consideration.

SECTION IV.

On Cometary Orbits, Aerolites and the Asteroids.

Believing that the law of gravitation is the law not only of the solar system but of the entire physical universe, we come now to the application of the circular theory to the movement of comets. As for the verifications of the elliptical theory of orbits by the observations and calculations of astronomers, I have at present simply to say that, since they have failed to discern the true form of planetary orbits which are exposed to their view entire, how can they feel so certain as to the form of that which they see only in part? The question is a question of demonstration. Feeling confident that the circular theory is sustained by proof in its own favor, and by disproof of the elliptical theory, it is not feared that much injury will be sustained by reason of the many so-called verifications.

With the following qualification we will proceed to apply the formulæ derived in the preceding demonstrations.

If the comets in their orbits about the sun are governed in their movement by the same law as the planets, their orbits are circular. This qualification is entered because we are not entirely certain that there may not be secondary laws of material import helping to govern the movements of celestial bodies. There might be stages involving forces of attraction and repulsion, in which event the course of a moving body might be greatly modified. But since the advocates of the ellipse proceed, assuming that planets and comets are subject to the same law, we will also proceed with that belief.

1st. Enck's comet. The perihelion of this comet is determined at 33,000,000. Let us first determine in what time and with what velocity this comet would revolve about the sun at the uniform distance of 33,000,000.

In $t^2 : t'^2 :: R^3 : R'^3$ let $t =$ time of the earth's orbit and R its radius, then $(1 \text{ yr.})^2 : (84 \text{ da.})^2 :: (95)^3 : (33)^3$.

By formula 4, non-eccentric vel. $\propto \frac{1}{\sqrt{R}}$ hence since the hourly

velocity of the earth = 68,000 miles, that of Enck's comet in the case supposed would be $\frac{1}{\sqrt{\frac{33}{95}}} = \frac{1}{.59}$ times 68,000 = 115,000 miles. That is, at the uniform distance of 33,000,000 miles from the sun, Enck's comet would perform a revolution about the sun in 84 days with an hourly velocity of 115,000 miles.

But since the actual time of Enck's revolution is $3\frac{1}{8}$ years, our proportion becomes $(1)^2 : (3\frac{1}{8})^2 :: (95)^3 : (212 -)^3$; that is, the radius of Enck's orbit is 212,000,000, which is 6.72 times the radius of its perihelion non-eccentric.

From formula 3, $A' \propto \sqrt{R} = \sqrt{6.41} = 2.5$, it appears that the area described in a unit of time in Enck's true orbit = 2.5 times the area of its perihelion non-eccentric. But at perihelion the radius vector of the true orbit is the same as the radius of the non-eccentric, hence the perihelion velocity in the true orbit is 2.5 times 115,000 = 287,000 miles per hour. At aphelion the radius vector being 11.82 times as great, the velocity at that point is 287,000 ÷ 11.82 = 34,000 miles per hour.

Since the radius of this orbit is 6.41 times its perihelion distance, its eccentricity is 5.41, and aphelion 11.82 times perihelion; that is, the radius of Enck's orbit is 6.41 times 33,000,000 = 212,000,000 —; aphelion distance = 391,000,000 miles.

By reference to Burritt's Geography of the Heavens, p. 256, it will be seen that our distances are exactly those there assigned to an ellipse whose eccentricity is 179,000,000 miles.

At this stage of development no doubt all of us expected a war about figures, but unexpectedly we arrive at the same results. How to account for this agreement is a question of itself. I hold and will show that the operations and calculations of astronomers in developing cometary orbits, are conducted in accordance with the circular theory, while they think they are dealing in ellipses, parabolas and hyperbolas. How is this suspicion corroborated? They take $\frac{1}{2}$ of the major axis of an ellipse as the *mean distance* from the sun, which becomes their index to the area described in a given time as well as the area and period of the orbit. Now, is it proper in the case of a narrow ellipse, whose eccentricity equals some six times its perihelion, that $\frac{1}{2}$ of the major axis is the mean distance? Has not the minor axis of the ellipse a share in fixing the mean dis-

tance from the focus? It can only be properly said of an ellipse that $\frac{1}{2}$ of the major axis is the mean distance when the minor axis is equal to the major, in which event the ellipse is a circle. We have here an elegant example of attaining to correct results by committing errors which compensate each other. The first mistake is in supposing that the ellipse satisfies the law of force tending to a fixed point. The second mistake is in accepting the doctrine of circles without knowing it. And since this is so clearly the case with Enck's comet, may we not suspect that our simple formulæ will satisfy and supercede the intricate calculations and verifications in which our astronomers feel so well fortified?

Until recently Biela's comet, whose present whereabouts is a subject of speculation, performed a revolution about the sun in 2,460 days, about $6\frac{3}{4}$ years, at a perihelion distance of about 83,000,000 miles.

Substituting in the general proportion $t^2 : t'^2 :: R^3 : R'^3$ we have $(1 \text{ yr.})^2 : (243 \text{ da.})^2 :: (95)^3 : (83)^3$, which expresses the time in which Biela's comet would describe a non-eccentric orbit of 83,000,000 miles at a uniform velocity found by formula 4 = 73,000 miles per hour.

But since the absolute period of Biela's comet was 6.7 + years, the general proportion becomes $(1)^2 : (6.7)^2 :: (95)^3 : (337)^3$.

From which: Aphelion = 589,000,000 miles.
Eccentricity = 254,000,000 miles.
Perihelion = 83,000,000 miles.
Radius = 337,000,000 miles.
Perihelion Velocity = 147,000 miles.
Aphelion Velocity = 21,000 miles.
Area in unit of time 2.02 times non-eccentric.

A comparison of these results with those derived by other methods again shows a remarkable degree of harmony. We proceed with an approximate value of perihelion, which is stated in Burritt at about 83,000,000, from which basis we differ but $\frac{1}{6}$ of 1 per cent. in aphelion, which may reasonably be attributed to a slight deviation from the true perihelion.

Faye's comet, with a perihelion distance of 161,000,000 miles, would complete a non-eccentric at that distance in 2.42 years, with

a uniform velocity of 52,000 miles per hour. Its period, however, being $7\frac{1}{2}$ years, the general proportion becomes $(1)^2 : (7\frac{1}{2})^2 :: (95)^3 : (3.82 \times 95 = 363 -)^3$.

Diameter of Faye's orbit	726,000,000	miles.
Aphelion " "	565,000,000	"
Eccentricity " "	202,000,000	"

Perihelion velocity, 72,000; aphelion velocity, 20,000 miles per hour.

The eccentricity of Faye's orbit is .556 by the elliptic standard, hence perihelion distance = .444 of radius. The formula for conversion to the circular standard being $\frac{e}{1-e}$, $\frac{.556}{.444} = 1.25 -$ is the eccentricity of Faye's orbit.

The non-eccentric orbit of Brorsen's perihelion requires a period of 189 days at the uniform velocity of 85,000 miles. The period of this comet being $5\frac{1}{2}$ years, the general proportion becomes $(1)^2 : (5\frac{1}{2})^2 :: (95)^3 : (3.11 \times 95 = 295)^3$.

Aphelion Brorsen =	538,000,000	miles.
Perihelion.....	62,000,000	"
Perihelion velocity..	121,000	"
Aphelion velocity...	14,000	"
Eccentricity.....	4.75	

Given, the period of D'Arrest's orbit 6.4 years and perihelion 111,000,000, we derive by the application of the same method:

Period of perihelion non-eccentric, $1\frac{1}{4}$ years.		
Velocity in non-eccentric....	63,000	miles per hour.
Aphelion distance.....	543,000,000	miles.
Perihelion velocity.....	107,000	"
Aphelion velocity.....	18,000	"
Eccentricity	1.94	

In this orbit the aphelion distance is about $\frac{1}{2}$ per cent. less than that given by Loomis. Since quite a small error in observation or calculation will account for such difference, there is no occasion for special comment.

The orbit of Pons or Winnecke is granted a perihelion of 73,000,000 and a period about 5.6 years.

The elements derived for its non-eccentric perihelion are:

Time	244 + days.
Velocity	59,000 miles per hour.
Aphelion	523,000,000 miles.
Perihelion velocity	218,000 "
Aphelion velocity	16,000 "
Eccentricity	3 —

Halley's comet is especially interesting as a sort of connecting link between the short term comets and those whose re-appearance has not yet been recognized, if indeed they ever do return.

The perihelion distance of Halley's is 55,700,000 miles, at which distance it would revolve in a non-eccentric orbit in 164¼ days at a velocity of 89,000 miles per hour.

The period of its revolution being 76.66 years, the elements of its orbit are obtained from the proportion $(1)^2 : (76.66)^2 :: (95)^3 : (18.04 \times 95 = 1713.80)^3$.

Aphelion	3,371,900,000 miles.
Perihelion velocity	694,000 miles per hour.
Aphelion velocity	8,000 " " "
Eccentricity	30 —

If Halley's comet be granted four times its perihelion velocity; that is, $4 \times 694{,}000 = 2{,}776{,}000$ miles per hour, it will either pass outside of the sphere of the sun's influence or return to perihelion in $8 \times 76 = 608$ years.

The dignity of the study of Astronomy as a means of culture is eloquently set forth by Prof. Rowland of Johns Hopkins University. To the student in general he says: "To train the powers of *observation* and *classification*, study natural history. Not only from books, but from prepared specimens and directly from nature; to acquire care in experiment and be convinced that nature forgives no error, enter the chemical laboratory; for training in exact and logical powers of reasoning, study mathematics; but to combine all this training in one, and to exhibit to your minds the most perfect and systematic method of discovering the exact laws of nature, study physics and astronomy, where *observation, common sense* and *mathematics* go hand in hand."

In the treatment of the orbits of long-term comets whose identity has not been determined, believing that they are governed in their

movement by the same law that regulates the bodies of known periods, it is necessary to believe also that they satisfy the law of equal areas and the general proportion $t^2 : t'^2 :: R^3 : R'^3$.

The comet of 1680, which approached within 574,000 miles of the sun at perihelion, is said to move in an orbit whose aphelion distance is 13,000,000,000. It was computed by Newton that this comet at perihelion moves with a velocity of 880,000 miles per hour. Others, since, have increased to 1,200,000 miles per hour. By an application of the rules and formulæ which furnish satisfactory results when applied to known orbits, it will be seen that all these estimates are far below the actual velocity at perihelion.

First, with what velocity and in what time will a body describe a non-eccentric of 574,000 miles radius?

From $t^2 : t'^2 :: R^3 : R'^3$ we derive $(365\frac{1}{4})^2 : (.17 +)^2 :: (95,000)^3 : (574)^3$.

Computing from the law which satisfies the relative and absolute planetary motions, it occurs that a body moving about the sun at the distance of 574,000 miles, will complete a revolution in .17 of a day, or about *4 hours*. The entire orbit of 574,000 miles radius is about 3,600,000 miles. Hence the uniform *velocity* with which a body must move to describe an orbit of 574,000 miles from the sun's centre, is about 880,000 miles per hour, and the *time* of revolution about $\frac{1}{6}$ of a day.

It will be seen from these results that the velocity allowed by Newton is simply the *non-eccentric* velocity of the comet's perihelion distance.

The necessary condition to an increased orbit in both time and size is an increase of centrifugal force, which is the same as an increase of velocity; the greater the increase of velocity, the greater the elements of the resulting orbit, and inversely, the greater the orbit in respect to time and mean distance, the greater the velocity necessary to satisfy the condition.

It has been computed, or estimated, that the aphelion of the comet of 1680 is 13,000,000,000 miles. If so, our general proportion $t^2 : t'^2 :: R^3 : R'^3$ becomes

$$(1)^2 : (555)^2 :: (95)^3 : (6,500)^3.$$

If, then, aphelion distance of the comet of 1680 is 13,000,000,000 miles, its periodic time is 555 years.

By formula 3, the *area described in a unit of time*, $a' \propto \sqrt{R}$. This law of variation is true in the comparison of all conceivable orbits. Comparing the non-eccentric orbit of 574,000 miles radius with that of 6,500,287,000 miles radius, we see that the latter radius is about 11,324 times that of the former, hence, the area described by the vector in the latter $= \sqrt{11,324} = 106$ times that in the former. At perihelion the radius vector of the orbit of the comet of 1680 is the same as that of the orbit with which it is compared, hence, their areas described in a unit of time vary as their velocities; that is, if the vector of the comet's orbit describes 106 times the area of that described by the orbit of 574,000 miles radius, its *velocity* at perihelion must be 106 times 880,000 miles = 93,000,000 miles per hour.

Here we may well stop a moment and demand an explanation of this staggering result,—a velocity covering the distance from the earth to the sun in an hour.

Having been led to this result by adherence to undisputed laws, we must be careful not to discredit simply because we can not comprehend. Admitting the uniformity of area described by the radius vector of an orbit, we must harmonize velocity in compliance with the demands of this law. We must agree, that, as the radius vector increases in length, velocity diminishes and *vice versa*. Nor must it be forgotten that mathematics furnishes a precision of results that defies the verification or contradiction of the astronomical observer, for it is a mathematical fact that the velocity just announced does not continue any appreciable length of time.

Should this rate of velocity be continued but for one second, (1,500,000 miles) since so small a section of so great a circle may be regarded as a right-angled triangle, the radius vector at the end of that second of time would be between 3 and 4 times its original length, demanding a decrease of velocity to $\frac{1}{3}$ or $\frac{1}{4}$ of its perihelion. At the distance of 45,000,000 from the sun (covering perhaps 10 or 15 times the diameter of the sun) this enormous velocity is brought back to about 1,000,000 miles per hour, beyond which its motion agrees more nearly with our power of conception. The aphelion distance of this comet being about 22,600 times that of perihelion, the velocity at aphelion must be about 4,000 miles per hour. When, therefore, we inquire carefully into the correctness of the results to

which we are led by undisputed laws; when we stop to reflect that the radius vector must describe the same area when almost nothing, as when it reaches thousands of millions of miles into space; when we question with what inconceivable momentum a body must be endowed to escape precipitation at so near an approach to the sun, we may appreciate the necessity of a velocity so far beyond our power of comprehension as that just announced for a body moving under the conditions *assumed* for the comet of 1680.

Modifications of orbits, by reason of the disturbing effect of planets upon each other, and the more serious deflections of comets have not entered into these demonstrations. Should the comet of 1680 have made so near an approach to the sun by reason of perturbations caused by planetary attraction, allowances in computing the elements of said orbit must be made in satisfaction of such disturbance. But if the course of said comet was determined almost exclusively by the action of the sun, the figures just given may safely be adhered to as being correct.

The effects which the action of Jupiter exerted upon the comet of 1770 emphasizes the importance of accounting for the interference of planets. If it can be shown that, aside from interference, the perihelion of the comet of 1680 amounted to the distance of Jupiter, by whose interference its course was so changed that it approached so near to the sun, then the figures for perihelion velocity must be modified in accordance with the facts. The qualification with which we end the application to the comet of 1680 is, that if said comet moves about the sun in an orbit whose perihelion is 574,000 and aphelion 13,000,000,000 miles, independent of perturbations, then the conclusions to which we have arrived are substantially correct.

When we remove our standpoint of demonstration to a greater perihelion distance, the results obtained are not so difficult of comprehension. For instance, the hourly velocity of Jupiter is about 30,000 miles, and the period of its orbit about 12 years. Should a comet pass its perihelion at the same distance as Jupiter with a velocity 4 times as great, or 120,000 miles per hour, the period of that comet would be $8 \times 12 = 96$ years. Knowing, then, the period and perihelion distance, or the perihelion distance and velocity, the orbit of any body moving about the sun may be readily determined by methods just concluded.

AEROLITES.

The precipitation of aerolites being intimately connected with other orbital movements, deserves an explanation in harmony with the new doctrine. It has been urged against the lunar origin of meteoric stones that the volcanoes of the moon are extinct. It will be shown how, possibly, these masses may have been hurled within the sphere of the earth's attraction at a period indefinitely remote, having since revolved about the earth in orbits of so great eccentricity that atmospheric resistance caused gradual contraction of the orbit, resulting in eventual precipitation.

There is a point between the earth and the moon at which their forces are equal, and a body placed in that point will be in a state of equilibrium.

Since the mass of the earth is 81 times that of the moon, $\frac{81}{r^2}$ expresses the force of the earth and $\frac{1}{r'^2}$ that of the moon at the required point. But $\frac{81}{r^2} = \frac{1}{r'^2}$ $\therefore r = 9r'$; that is, the point in question is 9 times as far from the earth as from the moon.. When the distance of the moon is 240,000 miles, the position of the point of equal attraction is 24,000 miles from the moon and 216,000 miles from the earth. If these masses of matter be hurled from the moon by a force sufficient to carry beyond 24,000 miles in the direction of the earth, they will be subject to the earth's attraction and independent of the moon except so far as perturbation.

It is evident that, if the force of expulsion is inadequate to carry the required distance, the masses will return to the moon.

Suppose E the position of the earth, m that of the moon in its orbit $m\,n\,o$, $P\,m = \frac{1}{9}\,P\,E$ and $E\,P' = \frac{9}{16}\,E\,m$. Suppose an eruption of the moon by which masses are hurled in the direction of the earth beyond the point P. Conceive the transfer to be instantaneous, then since P, and all points between P and E are nearer E than m is, it is evident from what has been determined

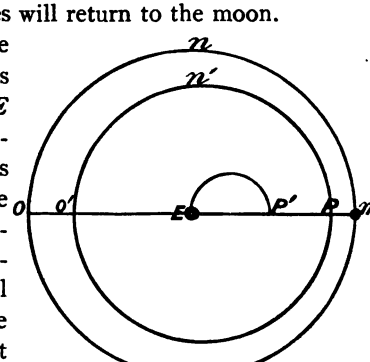


Fig. 24.

of orbital velocity that, in order to describe an orbit of equal eccentricity to that of the moon, the body at P requires a *greater* velocity than m. It is no less evident, however, that the body at P will have the same centrifugal velocity as m, in consequence of which the orbit will be more eccentric than that of m, and P is the farthest point from E. The relation of orbital velocity and distance, as determined from the law of gravitation and verified in the tables of planetary orbits, is, that in orbits of equal eccentricity the velocity varies inversely as the square root of the distance. Hence, at P', which is $\frac{9}{16}$ as far from E as m is, the velocity should be $\frac{4}{3}$ as great. But if we suppose a body instantaneously transferred from m to P', it will be possessed only of $\frac{3}{4}$ of the velocity of m, which is $\frac{1}{4}$ less than $\frac{4}{3}$, or the velocity required at P', to describe an orbit similar to that described at the distance $E\,m$. In connection with the theory of precipitation it was shown that, if a body describing a non-eccentric orbit be deprived of $\frac{1}{4}$ of its centrifugal motion, the consequence is immediate semi-circular precipitation. Taking into account the radius of the earth, precipitation will result at a distance somewhat more remote than P' from E. These points being determined we may advance the following general announcement;—if a particle of m be instantaneously transferred to a point between m and P, it will return again to m; if to a point between E and P', it will at once fall upon E; and if to any point from P to near P', it will describe an orbit about E.

Now the nature of the orbits described between P and P' is such, that the nearer P' the smaller and more eccentric the orbit, and the closer to E it approaches at its nearest. If near P' the orbit will pass through the denser atmosphere, which, by impeding progress, or, in other words, by reducing centrifugal motion, will cause precipitation in a comparatively few revolutions. As the point of translation recedes from P' toward P the tendency to precipitation naturally diminishes, and there is good reason to suppose that at distance P a body will describe a practicably permanent orbit about E with the velocity of m. Between P and P' then may be determined all the stages between immediate precipitation and an absolutely permanent orbit. By an application of the formulæ expressing the relations of the elements of orbits, the path pertaining to any point between P and P' may easily be determined.

According to these facts, the extinction of lunar volcanoes is no barrier to the theory that aerolites are of lunar origin, for we can easily account for the present precipitation of products emitted from its craters millions of years ago, if need be. The uniformly great density of aerolites is in harmony with the supposition that, in the event of any obstructing medium upon the surface of the moon, only the more compact forms of matter are carried beyond the sphere of attraction.

Nor are we required to conceive of an incredible force necessary to carry matter beyond the limit of the moon's attraction. It has been computed that projection with a velocity of seven miles per second perpendicularly will send a body beyond the limit of the earth's attraction. A velocity considerably less than this will transfer a body from the earth to the moon. By reasonable allowance for the difference of masses, it will require a velocity not greater than several miles per second to carry matter from the moon to the limit of the earth's control. Computations have shown that the effect of gunpowder reaches so high as one mile per second; hence, in the supposition that aerolites are lunar products we are required to conceive a volcanic capacity equal only to the effect of gunpowder; for the same force that generates a velocity of one mile per second on the surface of the earth will cause a velocity several times as great upon the surface of the moon.

The Asteroids.

Olber's hypothesis concerning the origin of the asteroids, has been objected to because the orbits of those bodies do not have a common point of intersection. It will be shown that no such point of intersection is needed, or indeed tolerated, in proving a common origin of the small planets. Had dismemberment taken place without the intervention of some great internal or external commotion, the demand for such point of intersection was just; but, since the assumption of a great explosion is the vital part of the hypothesis, the fragments may be so scattered that each will have a new orbit in all points distinct from the orbits of other fragments. Instead of detracting from its merits, this very circumstance gives strength to the supposition that the asteroids are fragments of a dismembered planet.

Suppose (Fig. 25) S the position of the sun and APP' the orbit of a planet. At A an explosion occurred which broke the planet into a multitude of fragments, scattering them in all possible directions. Each fragment is now endowed with the original velocity of the planet ± the effect of the explosion. Let $a\,b\,c$ and d be a few of the fragments. The force of the explosion being such that the several parts are carried to their respective positions, there being no considerable attraction between them, each will move in a new orbit of its own, and very soon they will be so widely separated that they will occupy all the relative positions from conjunction to opposition.

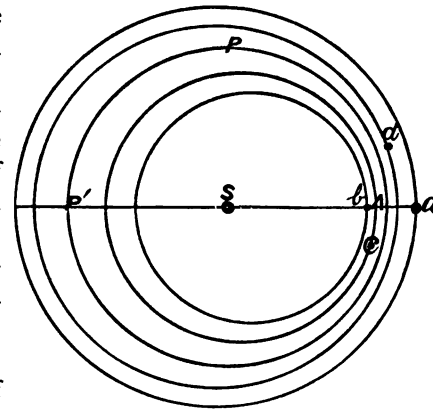


Fig. 25.

The fragment a, being displaced perpendicularly to the line of its motion, will move in its orbit with the same velocity as the original planet; but, being more remote from the sun, its velocity will be more than adequate to describe an orbit similar to the original. Its orbit will be *more eccentric* and its period *longer* than that of the original. The fragment b, being urged perpendicularly in the direction of the sun, requires a *greater* velocity than the original to describe an orbit of the same eccentricity. But its velocity being the same as that of the original, it will describe an orbit of negative eccentricity, of which b is aphelion. Its period will be shorter than that of the original.

The fragment d being hurled in the direction of the planet's motion has an accelerated velocity, causing a larger, more eccentric orbit of longer period than the original. The fragment c, being urged in a direction nearly opposite to the line of motion, its velocity is less than the original, by reason of which it pursues a path smaller and of shorter period than the original, negatively eccentric.

With this explanation, which agrees in every detail with the doctrine of circular orbits, by conceiving various directions and degrees of impulse resulting from the dismemberment of the original

planet, we may consistently account for all the various degrees of eccentricity, of inclination to the plane of the ecliptic, of periodic times, as well as for the noted entanglement of the orbits.

The incommensurability of their periods gives permanence to the orbits of the asteroids, but for which they would again accumulate in one common body. The acceptance of Olber's hypothesis, so agreeable to the new doctrine of orbits, affords us one of the sublimest chapters in the history of creation. Though insignificant separately by the side of a principal planet, the group or *train* of asteroids invites the imagination to a spectacle whose awful grandeur is unequaled by any other known convulsion of nature.

SECTION V.

Comparison of Theories and Criticisms Upon Newton's Demonstrations.

In conformity with the foregoing demonstrations it is asserted that if any body, whether planet, comet, meteor or satellite, describes an orbit about a central force, that orbit is circular, and the position of the central force with reference to the centre is determined by the ratio of centrifugal to centripetal force, and that any body once passing a perihelion point will, if undisturbed on its way, return again to the same point.

Although mathemathics is an exact science, it must not be supposed that fallacy is an absolute stranger in the domain of its research. The same care that is necessary in assuring the correctness of premises in inductive science must also be exercised in avoiding violence to axiomatic truths and general principles. As hasty generalization may lead to serious error in inductive science, so, also, may the mathematician be guilty of accepting approximation for correctness and likeness for identity.

In support of new theories which come in conflict with accepted doctrine, it is the reasonable duty of the innovator to bear the burden of disproof. In exact science this duty is especially binding, since only one of two or more conflicting conclusions can be tolerated. Compromising grounds and debatable conclusions belong only to induction. Modes of *reasoning* may be the same by induction as by demonstration, the chief distinction being in the *nature* of the *premises*, which in inductive science are at best but highly probable, while those of direct demonstration are absolute and unequivocal. The chief concern in inductive reasoning is a fit *selection* of premises—in deductive adherence to fixed laws and principles.

It is proposed to conduct the following strictures on the accepted doctrine of orbits in accordance with these demands, whereby, it is hoped, the merits of the old and the new theories may be more readily appreciated.

In tracing the history of the accepted theory of orbits it will be found, that in the earlier efforts observation was more conspicuously relied upon than demonstration. Originally the elliptical theory of orbits is simply the result of an effort to harmonize a few facts derived by observation. Once the mechanism of the sky was admitted to be as the martyred Galileo so firmly declared it—that the world moves, rather than the sun, there was opened to human research one of the most fruitful fields of discovery, and, we must not fail to add, speculation, that has ever engaged the mind. Nowhere is the omnipotence of the Deity more manifest; nowhere the grandeur and design of His works more evident; nowhere the absolute necessity of a first cause more positively demanded. As we walk in the pathway of science through nature "up to nature's God," while every new discovery affords us clearer apprehensions of His attributes, so also does every new fact discovered, every new phenomenon accounted for, more forcibly assure us of the comparative nothingness of self and all that is finite.

Returning to the theory of orbits: when the revolution of the earth about the sun was once conceded, the first and most natural assumption was that the orbit is a circle, of which the sun occupies the centre. By repeated observation it was found, however, that the apparent diameter of the sun varied from time to time. This circumstance could not be reconciled without a change of theory. The elliptical theory so well accounted for this variation that it was adopted, no doubt, without seriously inquiring whether the eccentric circle in which the sun is removed from the centre might not satisfy the same demand. Believing that the elliptic theory was correct, the next step was upon the discovery of the law of gravitation, to harmonize by demonstration the *law* with the *accepted theory*. It thus occurs that the conclusion is older than the demonstration. The object was not so much in this case to determine a truth as to verify a supposition. Kepler and Newton are jointly responsible for the elliptic theory of orbits, the first for framing, the latter for sustaining.

Long before Newton's time Kepler announced the following laws:

1st. That the radius vector of every planet describes about the sun equal areas in equal times.

2nd. The path of every planet is an ellipse, having the sun in one of its foci.

3rd. The squares of the times of revolution are the cubes of the mean distances from the sun, or as the cubes of the major axis of the orbits.

The seal to these facts (?) is that they are *revealed* by *observation.*

The laws of Kepler, in accordance with results upon which the circular theory of orbits is founded, are modified as follows:

1st. The radius vector of any body moving under the influence of central and centrifugal forces describes equal areas in equal times.

2nd. The form of every orbit is circular, the position of the central body depending upon the ratio of the forces.

3rd. The squares of the periodic times are as the cubes of the radii.

These laws, though not formally announced in the early part of this work, are fundamental, and all generalizations and formulæ conform to their strict interpretation. Before entering into a detailed examination of Newton's demonstrations it might be inquired: Does the theory of circular orbits fail to explain anything accounted for by the elliptical theory?

In the case of the long, narrow ellipse, it is explained that the *nearness* of the attracting force at perihelion accounts for the sudden curvature. But the symmetry of the curve demands the same sudden curvature when the body pursues the remotest part of its orbit. Will, then, the *presence* and the *absence* of said force account for the same effect? Granting that such movement is at all possible, it is evident that, while coursing along the flattened side, in which there is scarcely any curvature, centrifugal force is ascendant. How, then, can gravitation, which enfeebles as the square of the distance increases, cause the sudden curvature at aphelion? Is it not more reasonable that the *ratio* of the contending forces should be equal throughout the entire orbit, by reason of which the *curvature* is constant, as in the circular theory, than that the ratio should be the *same* for all *opposite* points and *different* for all points *contiguous*, as in the ellipse? Does it not appear more plausible that since the ellipse is a figure depending as much upon the existence of one focus as upon the other, that the only condition under which it can satisfy the law of gravitation is that equal forces should exist in either focus?

Figs. 26 and 27 are calculated impartially to illustrate the opposing theories of orbital motion. For evidence that Fig. 26 fairly represents Newton's conclusions, the reader is referred to Loomis' Treatise on Astronomy, pp. 138–141, or to Newton's own work. His announcement is as follows: If *S* be the seat of gravitation and *A* the position of a body at rest, if that body be allowed to fall freely to *S*, its velocity will be such that if allowed to move unobstructed it will pass to the opposite side of *S* to *B*, a distance equal to *A S*. After being brought to rest at *B* it will again return to *A* in condition to perform its vibrations indefinitely between *A* and *B*, If the force at *S* be conceived a *point* instead of a *mass*, this is the only condition in which Newton admits precipitation. So soon as the least assignable centrifugal force be applied to the body at *A* it will describe an ellipse so narrow as scarcely to be distinguished from a straight line. *S* represents the remote focus of the ellipse, the near focus being almost at *A*. The body advancing from *A* is now supposed to move down one of the flat sides of this narrow

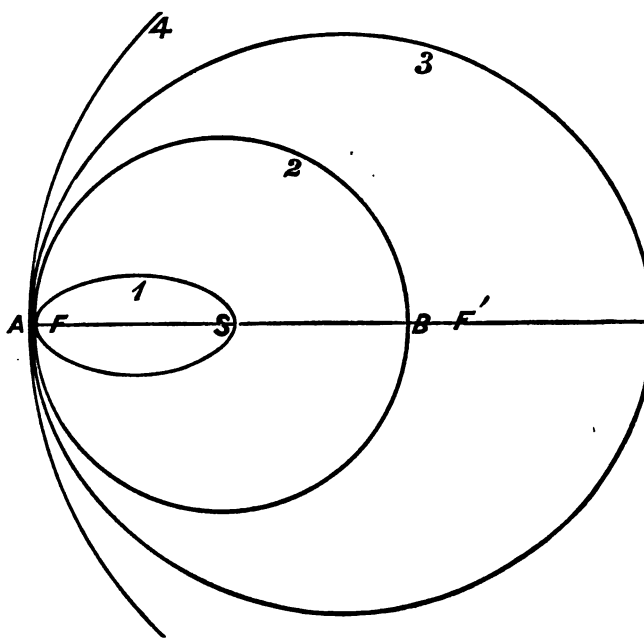


Fig. 26.

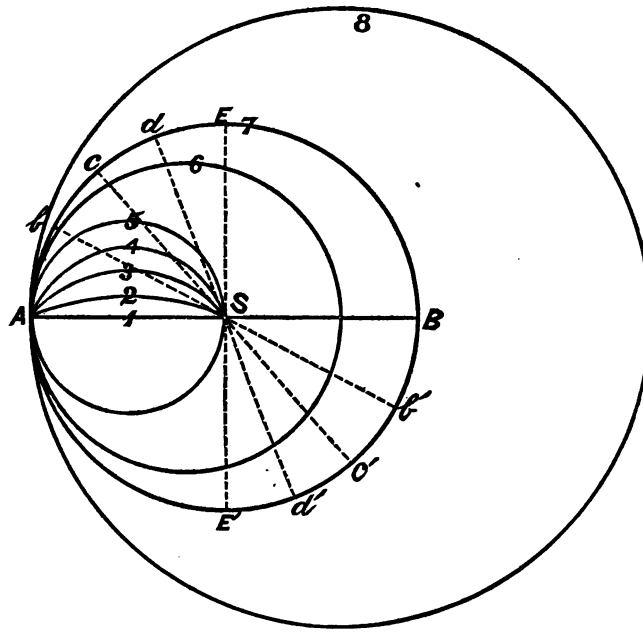


Fig. 27.

ellipse. Scarcely escaping contact with the point S, it wheels about in less than any assignable curve, returns to A on the opposite side of the ellipse in a permanent orbit.

No. 1, Fig. 26, represents an orbit of considerably greater centrifugal force than that just considered. S, the remote focus of the ellipse, is still near the extremity of the major axis. As centrifugal force at A increases, S remaining the remote focus of the orbit, the other focus F advances toward S. The eccentricity of the orbit diminishes and the ellipse becomes the circle No. 2, when centrifugal force is just adequate for such effect. It will be observed that this is the only condition in which Newton admits the *possibility* of a circular orbit. Just one particular rate of centrifugal force will produce this result, the slightest variation changing again to an ellipse. As the velocity of the moving member is increased beyond that required to describe a circle, the order of the foci is reversed, S is no longer the remote but the near focus, the other advancing upon the line $A\,B$ produced, the remote focus of No. 3 being F'.

As the velocity at *A* is indefinitely increased the ellipse becomes indefinitely enlarged and indefinitely attenuated, until *somewhere*, when centrifugal force is greatly increased, the moving member abandons the ellipse, adopts the parabola, and eventually even the hyberbola. If this theory of orbital motion were even correct, its want of a *law* to determine the *effect* of a *given increase* of centrifugal force upon the form and period of an orbit, reveals an imbecility aside of the circular theory illustrated by Fig. 27. No. 7 (Fig. 27) is the non-eccentric orbit corresponding with Newton's No. 2 (Fig. 26). It is the basis of comparison, and represents an equilibrium of centrifugal and centripetal force. As the centrifugal force of the moving member at *A* is diminished, the orbit contracts from No. 7 by gradation to No. 5, which is the limit of precipitation, representing 70 per cent. of non-eccentric velocity. 4, 3, 2 and 1 represent degrees of velocity diminished to a state of rest. *AB*, *b b'*, *c c'*, *d d'* and *e e'* represent the imaginary vibrations subsequent to precipitation, pertaining respectively to 1, 2, 3, 4 and 5. Velocity greater than that of No. 7 produces orbits of greater size and longer periods, depending upon an established relation between centrifugal and centripetal force.

Loomis, on p. 139 (Treatise on Astronomy), states a case deserving some attention. He says (by authority of Newton): that if a body at a given distance from the sun receives an impulse obliquely to the line connecting the body with the sun, the body may describe either an ellipse, a parabola or an hyperbola, but *not in a circle*. The possibility of a circle is admitted, when motion is imparted at right angles to the line connecting the body with the sun, but denied in the case of obliquity.

The *fact* is that, since the motion of a body describing an orbit may be supposed to be imparted to it in any part of the orbit, always in the direction of the tangent to that point, an examination of the various figures representing orbits, shows that the force of projection may vary in its angular relation to the radius vector, from 90° to 180°—the former at all aphelion and perihelion passages, and in the non-eccentric orbit constantly; the latter in the minimum orbit just at the time of precipitation. Situations may be selected to represent all intermediate degrees of inclination, any of which might be assumed as the position at which the original impulse was imparted. Perihelion and aphelion being more available in demonstration, velocity is assumed to originate there.

The conclusion from this fact is that the *form* of an orbit is the same whether the centrifugal impulse be imparted at right angles or obliquely. If imparted at right angles the point will be the future aphelion or perihelion of the orbit, and if obliquely an aphelion or perihelion will be determined.

In the case of an established orbit of given eccentricity, suppositions of accelleration will result, as follows:

If the velocity of a planet be increased at perihelion, the orbit will be larger, more eccentric and of longer period; if diminished, the orbit will be smaller, less eccentric or negatively, and its period shorter. An additional impulse at quadrature will increase the *size* of an orbit without changing eccentricity, while an increase at any other point will so alter an orbit that all its elements are changed.

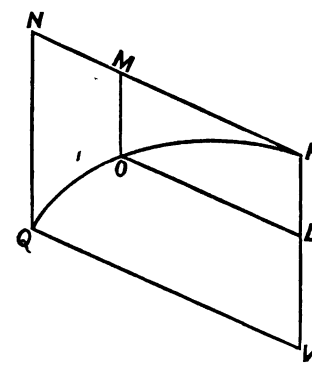


Fig. 28.

In Olmstead's Philosophy, by Snell, is a demonstration on the path of projectiles which, if adhered to, leads to mischievous results. Since the curvilinear motion caused by the influence of gravitation is the same, whether the projectile be a cannon ball or a planet, whether the path described be a circle or the millionth part of a degree, no demonstration should be based upon the supposition that gravity operates in parallel lines when it is quite as easy to apply the law of convergence. Following is the demonstration given: "Suppose a body to be thrown from P (Fig. 28) with an impulse which alone would carry it to N, in the same time as gravity alone would carry it to V. Complete the parallelogram $PNQV$; then, as both motions *co-exist*, the body will at the end of the time be found at Q. Let T be the time of describing PN or PV, and let t be the time of describing PM by the impulse or PL by gravity. Then, at the end of the time t, the body will be at O; now, as PN is described uniformly:

$PN : PM :: T : t$; $\therefore PN^2 : PM^2 :: T^2 : t^2$:. But, since the *effect* of gravity varies as the square of the time, $PV : PL :: T^2 t^2$ $\therefore PV : PL :: PN^2 : PM^2$; or $QV^2 : OL^2$. Hence the curve

is such that $PV \propto QV^2$; that is, the abscissa varies as the square of the ordinate, which is a quality of the parabola. POQ is therefore a parabola, one of whose diameters is PV, and the parameter to that diameter is $\frac{QV}{PV}$. Owing to the resistance of the atmosphere, the curve *actually* deviates sensibly from a parabola, especially in *swift* motions."

The more consistent treatment of this case, being in harmony with the *actual* manifestations of gravitation, is, that a particle being projected from P in the direction PN, is attracted by a force situate somewhere in the direction PV. Aside from the effect of atmospheric resistance the case is now identical with planetary movement, and the conclusion is evident that the path of a projectile is an arc of a circle. The change which results from the supposition that the force resides somewhere in the direction PV, instead of being in parallelism, is that the motion in the direction PM is no longer uniform. The force of attraction must be resolved into its deflecting and retarding factors, by reason of which the actual position of the projectile will evidently not describe the parabola as derived by the improper assumption of parallelism. The true position will be either outside and in advance of Q, or inside and behind—the former when the factor of gravitation accellerates; the latter if it retards, depending upon the position of the force in the line PV.

Having now presented the essential features of the circular theory of orbits, let us by way of indirect support inquire into the methods by which Newton arrived at opposing conclusions. In order that our criticism on his treatment of central force may be more intelligible, we present his series of propositions in full, as given by Frost.

Prop. 1. Theorem I. When a body revolves in an orbit, subject to the action of forces tending to a fixed point, the areas which it describes by radii drawn to the fixed centre of force are in one fixed plane, and are proportional to the times of describing them.

By this demonstration it is shown that Kepler's first law, that of equal areas in equal times, is true whatever be the law of force tending to the fixed point. It is all the same whether the force tending to the centre is *constant*, or varies *directly* as the distance or any

power of the distance, or whether it varies *inversely* as the square or any other power of the distance. The demonstration of Prop. 1 is clear and correct.

Prop. 2. Theorem II. Every body, which moves in any curve line described in a plane, and describes areas proportional to the times of describing them, about a point either fixed or moving uniformly in a straight line, by radii drawn to that point, is acted on by a centripetal force tending to the same point.

By the demonstration of Prop. 2 and its corollaries the fact is established, that when a body moving in a curve describes equal areas by radii drawn to a relatively fixed point, then the curvature is caused by a centripetal force residing in that point; and, if the areas described about that fixed point are not equal, the curvature is caused by a centripetal force *not* residing in that point.

The conclusion that is warranted by the results of these two demonstrations pertaining to orbital motion is that, if the planets or other bodies revolving about the sun describe equal areas in equal times, the curvature of their orbits is caused by a centripetal force residing in the centre of the sun; and conversely, if the bodies revolve about a centre of centripetal force, their radius vectors will describe equal areas in equal times. These propositions are true whatever be the law of centripetal force.

Prop. 3. Theorem. III. Every body, which describes areas proportional to the times of describing them by radii drawn to the centre of another body, which is moving in any manner whatever, is acted on by a force compounded of a centripetal force tending to that other body, and of the whole accelerating force which acts upon that other body.

Prop. 4. Theorem IV. The centripetal forces on equal bodies, which describe different circles with uniform velocity, tend to the centres of the circles, and are to each other as the squares of the arcs described in the same time, divided by the radii of the circles.

Prop. 5. Problem I. Having given the velocity with which a body is moving at any three points of a given orbit, described by it under the action of forces tending to a common centre, to find that centre.

Prop. 6. Theorem V. If a body revolve about a fixed centre of force, in any orbit whatever, in a non-resisting medium, and if, at the extremity of a very small arc, commencing from any point in

the orbit, a subtense of the angle of contact at that point be drawn parallel to the radius from that point to the centre of force, then the force at that point tending to the centre will be ultimately as the subtense directly, and the square of the time of describing the arc inversely.

The results of Newton's demonstrations up to this point are clear and correctly found, and all that is thus far determined applies equally to any law of force. There is nothing so far found that favors an elliptical orbit any more than a circular.

The next of Newton's demonstrations being remarkable, as the instance in which he failed to discover the true relation of central force to the circular orbit, will be given in full.

Prop. 7. Prob. II. A body moves in the circumference of a circle, to find the law of centripetal force tending to any given poin-in the plane of the circle.

Let $A\,P\,V$ be the circumference of the circle, S the given point to which the centripetal force tends, $P\,V$ the chord of the circle drawn through S from P, the position of the body at any time, and $V\,O\,A$ the diameter through V. Join $P\,A$, and draw $S\,Y$ perpendicular to $P\,Y$, the tangent to the curve at P.

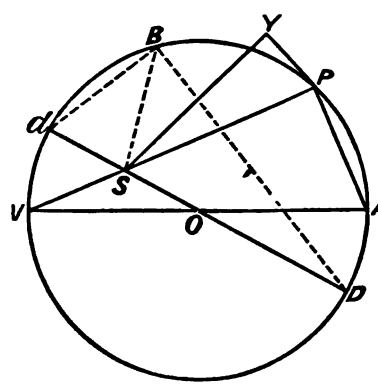


Fig. 29.

By Prop. 6, Cor. 3, if F be the measure of the accelerating effect of the centripetal force F, $= \frac{2\,h^2}{S\,Y^2\,P\,V}$, and since the angles $S\,P\,Y$, $V\,A\,P$ are equal, and also the right angles $P\,Y\,S$, $A\,P\,V$, the triangles $S\,P\,Y$, $V\,A\,P$ are similar, and $S\,Y : S\,P :: P\,V : V\,A \therefore F = \frac{2\,h^2\,V\,A^2}{S\,P^2\,P\,V^3}$; therefore, since h and $V\,A$ are given, F varies inversely as $S\,P^2\,P\,V^3$.

It is needless to trace this demonstration through all its reference, since the expression $F = \frac{2\,h^2\,V\,A^2}{S\,P^2\,P\,V^3}$, in which h represents twice the area described in a unit of time, is properly derived. By

reason of an oversight which Newton commits in the interpretation of the expression $F = \frac{2h^2 VA^2}{SP^2 PV^3}$, he failed to reveal the true form of orbits. There is hardly any supposition more plausible than that Newton entertained a preconception that the true form of orbits was elliptical, as declared by Kepler, and that, in his effort to prove it, he lost sight of true scientific methods. For once it appears that the great genius unwarily "worked for the answer."

In the course of the demonstration Newton says, "*let*" S and P be the positions of the central force and the moving body respectively. By so doing he *fixes* the positions of S & P, in consequence of which the chord PV being drawn through two *fixed* points is also a fixed, *i. e.*, *constant* chord. The supposition that the places of S and P may be changed, will not alter the given chord PV, for then the chord drawn through S from P will be an entirely other chord. The chord drawn in this latter case will also be a constant, as will indeed any chord of which two points are given. The infinite number of variations to the positions possible for S and P, will result in an infinite number of constant chords, and not in a single chord of an infinite number of variations, as is the case if PV is variable.

By applying this so rightful test, to the expression $F = \frac{2h^2 VA^2}{SP^2 PV^3}$, in which Newton recognizes the constancy of h and VA, but fails to acknowledge the equally evident constancy of PV, the final expression of variation is $F \propto \frac{1}{SP^2}$, which is the law of gravitation face to face. Had Newton not committed this oversight, he would have established by this very demonstration the theory of orbits, which it is proposed now to offer to the world. That SP is variable is evident from the definition of radius vector which SP represents. It must be observed also that SP has a double significance. As part of the chord PV it is constant; as vector, it is variable. Seeing that circular motion satisfies the law of gravitation, we are fully warranted in declaring that it is the true motion of the heavenly bodies.

The interpretation of Fig. 29, in accordance with these corrections, is that APV is the orbit of P revolving about S. Dd is the axis of the orbit, SO its eccentricity, d its perihelion and D its

aphelion. These conclusions are agreeable in every detail to the doctrine of circular orbits as developed in the present treatise. Were there no other evidence of the correctness of the circular theory, this examination of Newton's problem would seem to be sufficient.

The corollaries and observations on Prop. 7, though altogether faulty by reason of the error in the demonstration, need some attention, since the merits of the circular may be more clearly set forth thereby.

Cor. 1. "Hence, if the given point S, to which the centripetal force tends, be situated on the circumference of the circle, V will co-incide with S, and $F \propto \frac{1}{SP^5}$."

The fact is that in this corollary as well as in the demonstration, $F \propto \frac{1}{SP^2}$, the case here supposed being the limit of precipitation. Under the doctrine of circular orbits, this case arises when the ratio of centrifugal and centripetal force has reached its limit. An orbit is possible only so long as there is an assignable perihelion; that is, the centre of force must lie *within* the circumference.

Cor. 2. "The force, under the action of which a body P revolves in a circle PTV (Fig. 30), is to the force, under the action of which the same body P will revolve in the same circle, in the same periodic time, about any other centre of force R, as $RP^2 S P$ to SG^3, SG being a straight line drawn from the first centre S, parallel to the distance RP of the body from the second centre of force R, to meet PG, a tangent to the circle.

"For, by the construction of this proposition, since the periodic times are the same, the areas described in a unit of time are the same; therefore, h is the same for both centres; hence, if PRT be the chord through R, the force to S: force to R :: RP^2, $PT^3 : SP^2, PV^3$; but by similar triangles TPV, GSP, $PT : PV :: SP : SG$; therefore force to S : force to R :: $RP^2, SP^3 : SP^2, SG^3$ ::

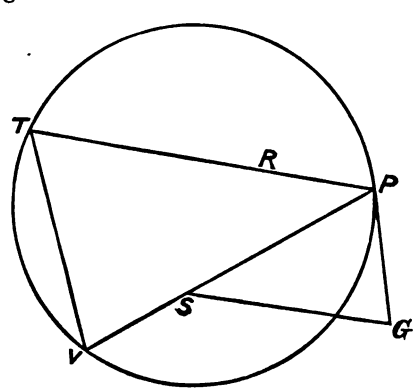


Fig. 30.

$RP^2\ SP : S\ G^3$." By applying to this corollary the correction made upon the demonstration, the proportion force to S : force to R :: RP^2, $PT^3 : SP^2\ PV^3$ becomes at once force to S : force to R :: $RP^2 : SP^2$. This is the true relation of the forces tending to the points S and R instead of that stated in the corollary.

By this result it is seen, that the force tending either to S or R, varies inversely as the square of the distance. It is therefore agreeable to assert that P will describe the orbit PTV in the same time, no matter what the position of the central force.

From, force to S : force to R :: $RP^2 : SP^2$, force to $S = \frac{RP^2, \text{ force to } R}{SP^2}$; and, since all these factors are variables. force to $S \propto \frac{RP^2 \text{ force to } R}{SP^2}$. This being true for all positions of R, suppose R in the centre of the circle, then RP and force to R, being both constant, force to $S \propto \frac{1}{SP^2}$, which is agreeable. In like manner it is shown that S being in the centre while R remains, force to $R \propto \frac{1}{RP^2}$. Assuming that both R and S be in the centre, then either SP or RP will be constant (*i. e.*, radius); hence force to S or $R \propto \frac{1}{1}$; *i. e.*, is constant. This conclusion is also agreeable, for in the non-eccentric orbit the force tending to the centre does not vary, since $\frac{1}{SP^2}$ is constant. Here it is important to distinguish between the *law* of force and the *exercise* of force. Though the force is constant in its *manifestation*, when exercised from the centre, the *law* of the force, that it varies inversely as the square of the distance, is still the same.

Corollary 3, together with a series of observation upon the demonstration of Prop. 7, are also subject to correction, so far as they partake of the fruits of Newton's oversight.

Prop. 8. Prob. III. A body moves in a semi-circle under the action of a force tending to a point S, so distant that the lines PS, QS, drawn from the body to that point, may be considered parallel; to find the law of force.

Let CA be a semi-diameter of the semi-circle drawn from the

centre perpendicular to the direction in which the force acts, cutting $P\,S$, $Q\,S$, in M and N, and join $C\,P$.

The demonstration of this proposition does not particularly interfere with the circular theory, but is obnoxious on account of its teaching that, under the special condition that the force is so distant, that $P\,S$ and $R\,S$ may be considered parallel, a body may move in *any* curve and still the law of force pertaining to S is the same, viz-: $F \propto \frac{1}{PM^3}$ for all

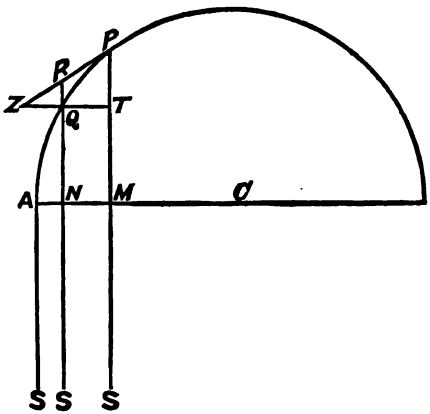


Fig. 31.

the curves. It is a cardinal principle in treating central forces that every curve has its *own* law. This principle commends itself to common sense, and is sustained by all properly conducted demonstrations. In our comment upon this demonstration, as on Theorem VII, we have more particularly to do with the result: Force at $P = \frac{h^2\,CP^2}{S\,P^2 PM^3} \propto \frac{1}{PM^3}$.

Newton does not say that $P\,S$ is infinite, but only so great that $P\,S$ and $R\,S$ may be considered parallel. It must be remembered that the value of the radius to the semi-circle may be many millions of miles, hence the value of $P\,M$ may also be millions of miles. $A\,S$, $N\,S$ and $M\,S$ are certainly equal. But, when Newton says that $P\,S$ is constant, he asserts that $A\,S = M\,S = M\,S + M\,P = P\,S$. This equality, if $M\,S$ is finite, surely violates the axiom that a *part* can not equal the *whole*. This Newton does when he says $S\,P$ is *finite* and constant. In truth, if $S\,P$ is *finite*, it is *not* constant, and Newton should say $F = \frac{h^2\,CP^2}{SP^2\,PM^3} \propto \frac{1}{SP^2\,PM^3}$.

Again, even this law of variation can not be maintained, unless the parallelism of $S\,P$ and $R\,S$ may be properly asserted, when S is fixed and $C\,P$ represents a great value. It is very evident that, if $A\,C$ is so great as to border on the infinite, it is inconsistent to

assert the parallelism of $A\,S$ and $S\,C$, $A\,S$ being perpendicular to $A\,C$. The unfitness of declaring parallelism between $A\,S$ and $C\,S$ varies as the ratio of $A\,C$ to $C\,S$,—most unfit when the ratio is nearest unity, least unfit when the ratio is nearest infinity, and fit only when infinity. By these arguments it is held that, if $S\,M$ is finite and $A\,C$ unlimited, it is improper either to assert the *constancy* of $S\,P$ or the parallelism of $S\,P$ with $S\,R$.

It might now be argued in defense, that the supposition that $S\,P$ is infinite will warrant equality and parallelism, and it may be that Newton means that $S\,P$ is infinite. Suppose it so, and what if force at $P = \frac{h^2\,C\,P^2}{S\,P^2\,P\,M^3} \propto \frac{1}{P\,M^3}$?

If $S\,P$ is infinite, as is necessary to warrant the parallelism and to defend the constancy of $S\,P$, the expression $\frac{h^2\,C\,P^2}{S\,P^2\,P\,M^3}$ becomes $\frac{h^2\,C\,P^2}{\infty,\,P\,M^3} = 0$, hence force at $P = 0$, and there is no law of variation at all. And what is more consistent than the conclusion that, if S is infinitely remote from P, it will exercise no influence at all, much less an influence governed by a fixed law?

Satisfied with the conclusion arrived at, Newton, adhering to the same method, concludes that an ellipse, a parabola or hyperbola may also be described under the same conditions and subjcct to the same law of force. If these conclusions of Newton were valid, we should believe that a body can under given conditions describe either of a number of curves at pleasure. We feel satisfied, however, that what has been shown is sufficient to prove that Newton has failed to find *anything* in this demonstration—that we have succeeded in showing that Prop. 8 should be promptly abandoned. The propositiou is an isolated one at best. It does not serve to prepare for anything following nor to connect other conclusions. Newton does not explain why he limits this curve to the semi-circle. How will the body move when at A? Since centrifugal and centripetal forces both tend in the direction of the central force, how can we escape the conclusion that the body will move in a straight line towards S, resulting in precipitation? Strike out Prop. 8, together with all the observations and examples, since when the methods are corrected it yields nothing that is available. It may appear somewhat remarkable that two consecutive conclu-

sions should be objected to; the first for failing to recognize the *constancy* of a certain factor, and the second for failing to notice the *variableness* of a factor. While these mistakes of Newton are somewhat similar in kind, there is an infinite difference in their tendency. The mistake committed in Prop. 8 is strictly local, while that in Prop. 7 has resulted in a false doctrine of orbits, thereby being perhaps more far-reaching in its pernicious influence than any other known mathematical error.

Prop. 9, Problem IV, proves that the law of centripetal force tending to the pole of an equiangular spiral, is $F \propto \frac{1}{SP^3}$. This is the same law as was erroneously determined in Prop. 8.

Prop. 10, Prob. V, shows, that if a body moves in an ellipse about a force tending to the centre, that force varies as the distance. In connection with this law which, it is claimed, is the only law of the ellipse with a single central force, it is determined that the periodic time is the same for all degrees of eccentricity —that the most eccentric ellipse and a circle whose diameter equals the major axis, will be described in the same time. This is the law of force existing within the earth where all imaginary movements comply with it.

Before examining Prop. 11, let us indulge in a little digression. The invention of the calculus has introduced the possibility of corresponding errors, against which the mathematician must carefully guard. There is danger of supposing that the ratios of finite quantities are lost in the ultimate. For example, suppose a right-angled triangle whose sides are $x. y. z$. Reducing the *size* of the triangle with unchanged angles, it might be supposed that, when so small that the area is less than any assignable value, the sides reduce separately to 0 whence their ratios are indeterminate as might appear by the form $\frac{0}{0}$. It must not be forgotten, however, that *only* the *sides* vanish. The *angles*, and consequently the *ratios* of the sides, remain not only *to* the ultimate but *in* the ultimate as well. Regarded in this light the mathematical *point* remains invested with all the vital properties of the finite form reduced to its ultimate. Thus conceived, the point is to *space* what the *atom* is to matter. By investing the point with a formula there will grow

the corresponding finite form represented by that formula. This law of development is analogous to the *life* principle which bestows characteristic upon the fundamental atoms of animate existence. Differential values, though represented by 0 must not be mistaken for absolute naught. Though a differential escapes our power of conception it is nevertheless a reality, a value. Higher powers of intellect than human can no doubt deal with the infinite as we manipulate finite values. Although we cannot grasp the infinite it may be fairly illustrated that infinites are liable to increase and decrease. As an illustration conceive any circle. Select a point within that circle. It is plain that an infinite number of chords may be drawn through that point. Our understanding will hardly tolerate us to think otherwise than that a greater number of chords can be drawn through two points than through one, and since there are an infinite number of points within a circle, through each of which an infinite number of chords may be drawn, how can we escape believing that relations exist among the infinites as well as among the finites, though our capacity is not sufficient to discern? Grave errors may be committed, by supposing even that the smallest side of a triangle, whose angles are maintained, will vanish before the other sides. So small a mistake being committed by Newton, in his demonstration of Prop. 11, gave rise to a false conclusion, whose validity appears never to have been disputed.

We have said that, although the human mind is unable to manage the infinites, there is reason to believe that the infinite is capable of variation. We assert (Fig. 31) the infinity and equality of AS, NS and MS. When PM represents a very great number, although $MS + PM = \infty$, it does not appear proper to say $MS + PM = MS$. Or how does it satisfy our comprehension to think that $AS + NS + MS$ which $= \infty = AS$ itself $= \infty$. Would it not appear that the nearest limit of the infinite is the farthest limit of the finite, and that the infinite is liable to variations within its sphere? Does it not commend to our sense to suppose that the infinites are units to the God-mind, to whom our finites are but decimals?

Thus much out of the way let us turn to an examination of Newton's Prop. 11.

Prop. 11. Prob. VI. A body is revolving in an ellipse, to find the law of force tending to a focus of the ellipse.

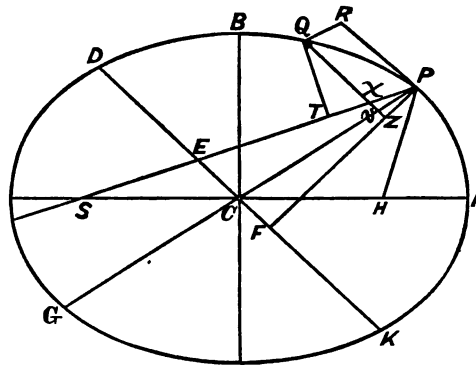


Fig. 32.

Let S be the focus to which the force tends, P the position of the body at any time, PCG, DCK conjugate diameters, Q a point near P, QT, PF perpendiculars on SP, DCK, from Q, P respectively, PR a tangent at P, QR parallel to SP, Qxv parallel to PR, meeting SP in x, and PC in v, and let SP, DCK intersect in E.

Then $F = \frac{2h^2}{SP^2} \cdot \frac{QR}{QT^2}$, ultimately, when PQ is indefinitely diminished.

But by similar triangles QTx, PFE:

$$\frac{QT^2}{Qx^2} = \frac{PF^2}{PE^2} = \frac{PF^2}{AC^2} = \frac{BC^2}{CD^2}.$$

Now $\frac{Qv^2}{Pv, vG} = \frac{CD^2}{CP^2}$, by properties of ellipse, and $\frac{Pv}{QR} = \frac{Pv}{Px}$ $= \frac{CP}{PE}$, by similar triangles,

$$\therefore \frac{Qv^2}{QR, vG} = \frac{CD^2}{CP, AC},$$

and $vG = 2CP$, $Qx = Qv$, ultimately;

$$\therefore \frac{QT^2}{QR} = \frac{2BC^2}{AC} = L \text{ ultimately,}$$

if L be the latus rectum of the ellipse;

$$\therefore F = \frac{2h^2}{L} \cdot \frac{1}{SP^2} \propto \frac{1}{SP^2}.$$

By this conclusion, if it be sustained, Newton proves that an elliptical orbit with the force in a focus satisfies the law of gravitation. Let us inquire into the method of demonstration.

Were it not for the fact that more and clearer evidence is demanded in the disproof of an accepted proposition than in the establishment of a new one, a simple announcement of the exception to Newton's work would appear to be sufficient. As it is, we will inquire carefully into the validity of the expression $Qx = Qv$.

Produce vx to z, then will QTx, Pzx be similar triangles of which Qx, Px are corresponding sides. As the point Q recedes from P, it will be observed that the triangle Pzx increases more rapidly than the triangle QTx. When Q is at B the triangles are about equal. Conversely, as the point Q approaches P from B, the triangle Pzx will diminish more rapidly than QTx. It might appear from this circumstance that upon the approach of Q upon P the triangle Pzx would vanish while QTx is finite. It will be observed, however, that as x approaches P, both Qx and Px approach 0; that Qx is finite only so long as Px is finite; that Qx and Px will vanish simultaneously, whence the triangles QTx, Pzx vanish simultaneously.

Now, since vx is a part of zx, vx will also co-exist with zx. In other words, all the values that pertain to the triangles QTx and Pzx are co-existent.

Finitely, it is clearly a violation of the axiom, that a whole is greater than any of its parts, to say $Qx = Qv$. Ultimately, as was shown, $Qx = 0$, and the expression $Qx = Qv$ can only be interpreted $0 = 0$.

The substitution of these values in former equations containing finite values, results in a conclusion containing 0 in disguise.

Let us illustrate:

$$\frac{QT^2}{Qx^2} = \frac{BC^2}{CD^2} \text{ and } \frac{Qv^2}{QR, vG} = \frac{CD^2}{CP, AC},$$

$$\frac{QT^2}{Qx^2}. \quad \frac{Qv^2 \text{ (now } Qx^2)}{QR, vG} = \frac{BC^2}{CD^2}, \frac{CD^2}{CP, AC};$$

$$\frac{QT^2}{QR, 2CP} = \frac{BC^2}{CP, AC}; \frac{QT^2}{QR} = \frac{2BC^2}{AC}.$$

These operations include the missing links of Newton's operation. We have shown that ultimately $Qx = 0$, hence the form $\frac{QT^2}{Qx^2 \text{ (ult.)}} \times \frac{Qv^2 \text{ (ult.)}}{QK, vG} = \frac{QT^2}{QR, 2CP}$ is simply $\frac{0}{0}$ in disguise.

Then we have $\frac{2BC^2}{AC} = \frac{0}{0}$; that is, the latus rectum of the ellipse is indeterminate, all on account of the unwarranted equality of Qx and Qv. The conclusion of it all is that, without this false equation, Newton can not derive any law of force applicable to the focus of an ellipse, much less prove that that law is the law of inverse square. Look at the expressions, $vG = 2CP$, ult., and $Qx = Qv$, ult. The first of these commends itself without remark.

The ultimate equality of the arc, the sin. and the chord is also well established, but the equality of Qx and Qv has no better argument in its favor than that, ultimately, the difference between the two is less than an assignable value. By such reasoning it might be proven that in an infinitely small circle the diameter equals radius, since the difference is less than an assignable quantity.

It is not only impossible to prove that the force tending to the focus of an ellipse varies inversely as the square of the distance, but it is impossible to show that an ellipse can at all be described under a *single* focal force. It requires a force in each focus. It is no more true that $Qx = Qv$ than that $Qx = QT$ or Tx or $QT + Tx$.

There is a special condition by which the equality of Qx and Qv may be properly asserted. It is a condition by which vx vanishes, while Qx is finite.

Conceive in Fig. 32 that S is moving in the direction SC. As S approaches C, vx diminishes. When S is at C, vx has disappeared and then truly $Qx = Qv$. Then it will very properly result, even with Fig. 32, that the force tending to S varies inversely as SP^2. But what is the dilemma? The correctness of the result depends on the supposition that S is in C, but when S is in C the ellipse is a circle. By reasoning with such unallowable premises as $Qx = Qv$, it is possible only to produce impossibles and contradictions. It is no less violent to say that *ultimately* all the sides of any polygon are equal; ultimately the major and minor axes of the most eccentric ellipse are equal; ultimately there is no difference between the whole and the greater part of an object, etc., etc.

Prop. 12. Prob. VII. A body is revolving in a hyperbola, to find the law of force tending to the focus of the figure.

The demonstration begins by stating that "the investigation is exactly the same as in the last proposition, employing the subjoined figure" (a hyperbola).

We may also say that the criticism of this demonstration being the same as that of Prop. 11 it is needless to repeat.

The conclusion that the force tending to the focus of the hyperbola varies inversely as the square of the distance, assumes the correctness of the corollaries to Prop. 7, and the misleading expression $Qx = Qv$.

Newton says, "in the same manner as in these propositions (11, 12) it can be shown that the *repulsive* force tending from a focus, under the action of which the body describes the opposite branch of the hyperbola, varies in versely as the square of the distance." To which again we have a right to reply that this conclusion, which depends upon the correctness of the former, is like them false.

Prop. 13. Prob. VIII, is demonstrated to show, that if a body moves in a parabola under a force tending to the focus, that force varies inversely as the square of the distance.

The same objectionable assumption is made that $Qx = Qv$, so that the merits of the conic trio as applied to central force rest upon a common claim.

Granted that the ellipse must yield to the circle there is no apology for the parabola or the hyperbola.

At the conclusion of these three demonstrations, Newton says, "it follows from the last three propositions, that if any body move from the point P in any direction PR, with any velocity, and be at the same time acted on by a centripetal force, which varies inversely as the square of the distance, the body will move in some one of the conic sections, having a focus in the center of force, and conversely.

For, when the focus, the point of contact, and the position of the tangent are given, a conic section can be described which will have a given curvature at that point; but, when the force is given and the velocity of the body, the curvature is known; and *two* orbits *touching* one another cannot be described with the *same centripetal force*, and the same velocity at the *point of contact*.

If the velocity, with which a body leaves its position P, be such that the body would describe the small space PR in some very small time, and in the same time the centripetal force were able to move the same body through the space RQ, this body will move

in some conic section whose latus rectum is the limit of $\frac{QT^2}{QR}$ when the lines PR, QR are indefinitely diminished.

In these corollaries the circle is included as a particular case of an ellipse; and the case is *excluded* in which the body moves in a straight line to the centre of force."

In the course of these remarks, Newton clearly states that when the conditions of centripetal force and velocity are given, only *one* resulting orbit is possible; that if a body moving from a given place describes an arc of any curve in a given small time, it can not under the same conditions describe the arc of any other kind of curve; that, though motion in all the conic sections is possible each curve has its own set of conditions. Newton, does not, however furnish means to determine by what change of conditions the ellipse becomes a circle, the circle an ellipse, and the ellipse, a parabola and hyperbola. He only says by *increased* velocity.

By the new doctrine of orbits it is also contended, that under given conditions only *one* form of orbit is possible, and that the properties of the sector described in some small time, determine the properties of the orbit. It has been shown by various demonstrations that, for all possible rates of velocity, movements controlled by the law of inverse square are circular; that a change in the rate of motion affects the size and eccentricity of an orbit, but not its form. The circular theory gains eminent advantage by supplying a set of formulæ, representing the relations of mean distance, periodic time, perihelion velocity, eccentricity, etc.

It is further claimed that all of Newton's demonstrations when properly conducted, endorse the circular theory of orbits, as has been shown by the various criticisms.

As a final remark upon his treatment of the conic sections, Newton says: "the circle is included as a particular case of the ellipse, but the case is *excluded in which the body moves in a straight line to the center of force.*"

Here Newton evidently leads himself into inextricable difficulties. By his theory, as the centrifugal velocity diminishes, the ellipse narrows, hence the focus approaches the extremity of the major axis. When the velocity is very small, the focus is so near the extremity that the body curves suddenly about the central force, and returns by the opposite side of the ellipse. If Newton does

not exclude the case of motion from rest, he is forced to admit that, falling from rest, the body will advance to the centre of force and, without passing beyond, will be invested with an inverse motion by which it may return to its original position. This is the conclusion to which Newton's theory forces you. He sees that this conclusion is absurd, so he simply evades the case which leads to it. It is strange that the absurdity of motion in the very narrow ellipse, which is almost equally evident, should escape the keenness of Newton's judgment. All admit that, falling from a state of rest, a body will pass an equal distance beyond the centre of force. Accepting this fact who can believe that, when endowed with the most trifling centrifugal momentum, the body should almost momentarily reverse its motion, although, in compliance with the law of equal areas, moving with inconceivable velocity? By Newton's theory this difficulty confronts you, whether you try to harmonize motion from rest with the doctrine of orbits, or the doctrine of orbits with the accepted theory of motion from rest.

Newton conducts his investigations with remarkable silence about precipitation. I do not distinctly remember him using the term. He appears to be aware of the fact that a consistent allowance for precipitation is fatal to the narrow ellipse, and hence to the entire doctrine. There is no argument that finds more ready acknowledgment, when rightly understood, than that which supports the theory of precipitation; that, when centrifugal motion is diminished to a certain limit, the body will pass to the center of force in a path whose curvature depends upon the remnant of centrifugal force. The conclusions arrived at by the circular theory, require no evasion of the case in which the body falls from a state of rest. This is but the *limit* of reduced centrifugal motion. The merits of the contending theories with reference to precipitation, have already been considered in connection with Fig. (25–6).

Astronomers have concluded that certain comets move in parabolic orbits. It is unquestionably more owing to Newton's demonstrations, than to convictions resulting from tests by observation, that such conclusions are entertained. The existing theory of orbits rests more especially upon the results of Newton's demonstrations than upon Kepler's observations. Take away the support afforded by Newton, and let us examine how far the true form of orbits can be determined by observation.

Kepler accepted the ellipse because it accounted for the observed changes in the apparent diameter of the sun. He tried the circle, supposing the sun in the centre. Between it and the ellipse the latter is decidedly the better choice. The circle, which represents the new theory, takes the middle of the supposed ellipse for its centre instead of the sun. The circle and the ellipse co-incide at aphelion and perihelion. In the case of all slightly eccentric orbits the curves of the two theories so nearly co-incide throughout, that the nicest measurements of the sun's apparent diameter will not enable the observer to decide between them. Take the orbit of the earth as an available illustration, its eccentricity being about $\frac{1}{60}$. Construct an ellipse of the given eccentricity, and with its major axis as diameter construct also a circle. These curves will both satisfy the variations of the sun's apparent diameter, and they will so nearly coincide throughout that the observer can not decide between them. Observation can settle the question of form between two hypotheses which greatly differ, but when they so nearly agree, as in the present instance, the observer is helpless to a great extent.

It remains for mathematical analysis to make the final selection. Observations conducted with reference to these two forms of orbits, as applied to planets, will scarcely ever differ beyond a few seconds. It is not even claimed that the elliptical orbit represents the place of a planet exactly, even allowing for well-computed perturbations.

Prop. 14. Th. VI of Newton, based upon the correctness of preceding demonstrations, reads: "If any number of bodies revolve about a common centre and the centripetal force vary inversely as the square of the distance, the latera recta of the orbits described, will be in the duplicate ratio of the areas, which the bodies will describe in the same time by radii drawn to the centre of force."

It is a self-evident conclusion, that if the bodies revolving about a central force do not move in either the ellipse, the parabola, or the hyperbola, the relations announced in Prop. 14 can not be correct.

Formula 3, page 30, is $a' \propto \sqrt{R}$, in which a' represents the area described in a unit of time and R the radius of the orbit.

Squaring both sides $a'^2 \propto R$. Altering Newton's Prop. 14, in conformity with this formula, it will read: if any number of bodies revolve about a common centre, and the centri-

petal force vary inversely as the square of the distance, the radii of the orbits described will be in the duplicate ratio of the areas, which the bodies will describe in the same time by radii drawn to the centre of force.

Prop. 15. Th. VII. On the same supposition, the squares of the periodic time, in ellipses are proportional to the cubes of the major axis.

By Theorem IX, of the circular theory, we have already demonstrated that, *if two or more bodies revolve in circular orbits, the squares of the periodic times will vary as the cubes of the radii.*

THEOREM XV.

(OF THE CIRCULAR THEORY.)

If any number of bodies revolve about a common centre of force and the centripetal force vary inversely as the square of the distance, $\sqrt{(a+e)^2 - e^2}$ *represents the distance at which the several orbits cut the axis of ordinates, referred to the central force, a being perihelion distance and e the eccentricity of any orbit.*

For, if a is perihelion distance and e the eccentricity, $a + e = R$, or radius of the orbit. The radius of the orbit forms a right angled triangle with the axis of ordinates and eccentricity. If y is the distance from the origin at which the orbit cuts the axis of ordinates, we derive, $y = \sqrt{R^2 - e^2} = \sqrt{(a+e)^2 - e^2}$.

Since e is a variable, depending upon a and the *velocity* of the body, this formula may be interpreted to the utmost generality, representing all possible rates of velocity for the same perihelion, and all possible changes of both speed and perihelion distance.

The superiority of the new doctrine of orbits is, perhaps, nowhere more decided than in the formulæ which established a general relation between central force, perihelion distance, velocity and eccentricity. According to the old doctrine, we have no measure to determine the variation of velocity required to produce a given change of eccentricity. No one can tell the gradation by which the ellipse becomes a circle, or at what limit the ellipse gives way to the parabola, and the parabola to the hyperbola, while by the new doctrine, central force, perihelion distance and velocity, or eccentricity constitute a trio so intimately connected that any two being

given the other may readily be found. So long as either term of the trio is finite, all are finite. The definite relations existing among the elements of the circular orbit, warrant the assertion, that if any body, planet, comet, meteor, or whatever passes a perihelion under the influence of the central force alone, it will, if undisturbed on its way, return sooner or later to the same position. The qualifying conditions, which will interfere with a prompt return, are: 1st, the perturbations arising from the attractive forces of other bodies encountered on the way; 2nd, the resisting media through which a body may happen to pass; and 3rd, if any body, by a force exerted on it outside of the solar system, is propelled with sufficient force in the direction of our system, to come within the sphere of the sun's dominion, the body will pass a perihelion with sufficient velocity to carry it outside the system, in which case it will not return. It does not follow, that because the body in the last case does not return, that the track described within the solar system is part of an infinite circle. It is finite, and the body would return but for the interference of other systems of the universe. If said body be sent within the limits of solar attraction, in the direction of the sun's centre, it will like any other body deprived of centrifugal motion, fall into the sun.

With these facts as data, it may reasonably be conjectured from the eccentric movement of comets, that they are bodies not primarily belonging to the solar system, but brought within by the motion of our system, or by external commotion, or by both. Those comets which are not materially disturbed by causes other than the sun's attraction, will pass again beyond the limits of the system; while, if their momentum is sufficiently diminished by the influence of the planets, they will remain within the system.

The sun alone will in no case cause a comet to remain within the sphere of its influence, because the aggregate of its accelerating effect, while the comet approaches it, is just equal to the retarding effect while the comet recedes, in consequence of which the comet will disappear from the solar system with the same velocity as it entered.

But an arrangement of the greater planets may readily be conceived, by which, while they insensibly affect the comet in its approach, they quite materially interfere with its departure. The reverse of this supposition is also to the point, and is sustained by actual occurrence.

The condition upon which a comet remains within the solar system, is that planetary interference deprives the comet of the amount of momentum with which it entered the system, or rather, a *little more* than the momentum with which it entered.

The perihelion distance of a comet, by this hypothesis, depends upon the angle of entrance to the solar system, its movements, when once confined to the system, being subject to all the laws and privileges of natives-born. The comet, if this supposition be true, is a foreigner by birth, coming within the dominion of our day-king as a stranger, becoming a naturalized member when its periodic time is determined. It was not the purpose of the author of these discussions to venture an opinion concerning the nativity of the comet. The conclusions just given are natural suggestions however, and may be accepted or rejected without interfering with the object of this work.

An examination of the various supposable laws of gravitation, reveals the peculiar fitness of the law of inverse square as the regulator of cellestial movement.

In the ellipse, whose force is in the centre, gravitation varies as the distance, being greatest when the revolving body is farthest, and *vice versa.* When the body is at the extremity of the major axis, the central force being at its strongest, is more than adequate for the centrifugal force, causing the sudden curvature at that point.

Very soon the force tends greatly to accelerate the velocity of the moving body. At the extremity of the minor axis the velocity is more than adequate for the now diminished central force, by reason of which the body recedes from the centre until the force attains its high tension at the other extremity of the major axis, after which the former movement is repeated. In the orbit, whose central force varies as the distance, there are two maxima and minima of force and velocity in each revolution. Contrary to actual planetary movement, the velocity in the ellipse, whose force tends to the centre is greatest when the force is least and, *vice versa.* The radius vector, even in this orbit, describes equal areas in equal times.

But what would be the consequence of such a law of force applied to the motion of celestial bodies? Since the force increases with the distance, those bodies will be most disturbed which are at the greatest distance; *i. e.*, if such a law prevailed, the fixed stars

would be more exposed to disturbance from our sun than would be our planets, thus rendering the commotion so notorious that the economy of such a law is not worthy of farther consideration. Forces varying as higher powers of the distance, are subject to objections still greater.

If the law of force were such that variations were *inversely* as the distance, the existing harmony of motion could not be maintained. A much greater centrifugal velocity would be necessary to maintain equilibrium, greater inter-planetary spaces would be needed to prevent fatal perturbation, and a closer nestling of secondaries to their primaries would be needed to resist the force of the sun.

The existing law of gravitation is a wise selection of means to an end. By this law the forces are sufficiently potent to control the movements of a system, without deranging the harmony of adjacent systems, or the individuals of the same system. Controlled by this law the regions of space can contain a much greater number of bodies than by a law of less restriction.

A law of force varying inversely as a higher power of the distance, would fail to readjust eccentric movements, and would render the remoter planets more subject to each other, while in conjunction, than to the sun, by reason of which they would revolve about each other. It has been shown by demonstration (Newton's Prop. 9) that, if a body moves about a central force varying inversely as the cube of the distance, the path is an equiangular spiral.* The inevitable consequence by such a law is precipitation by the equiangular spiral. The symmetry of movement, by the law of inverse square, is owing to the retarding and accelerating factor of central force, which is capable of restoring the lost equilibrium.

By the accepted doctrine of orbits, the law of gravitation applies to elliptical movement of all degrees of eccentricity, hence it applies when eccentricity vanishes and the ellipse becomes a circle, but then forsakes the circle. By the new doctrine, the application of the law of gravitation belongs only to the circle, allowing the central force to occupy any position within the circumference.

This new doctrine, as will be observed, is so entirely hostile to the old that there is no room for compromise.

Seeing how nearly the phenomena of planetary movement have been accounted for under the supposition of ellipticity, remember-

*See Art. 3 and 4, Appendix.

ing what bold predictions concerning the return of comets have been verified, it might appear as the height of presumption to question the correctness of a theory so satisfactory and so ably supported. The firm conviction, however, that the new theory will fail in nothing wherein the old succeeds, and succeeds wherein the old has failed; that it is more far-reaching yet more easy of comprehension, treating cometary and all other extraordinary movements with the same facility as planetary motion; that it is supported at every stage by faultless demonstration—in short, that it is the true doctrine of orbital motion, and will conduce largely to the advancement of Astronomical science—these are the reasons for presenting this work to the public. An apology for this audacity may be found in Newton's own remark, who replied, when a friend intimated that he left nothing for future discovery in the field of his investigations, "shake the bushes well and you shall have plenty of game."

When the comparison of the two theories is once fully instituted, the infirmity of the elliptical manifests itself at every stage. One of the most striking advantages of the circular theory is found by comparison of the following expressions. According to the circular theory $T^2 \propto R^3$, while by the elliptical theory $T^2 \propto A^3$.

By the circular theory we are enabled to derive a set of formulæ expressing fixed relations between the elements of any orbit, which can not be done in the case of the ellipse; for, since the square of the periodic times varies as the cubes of the semi-major axes, the minor axis having no part in determining the *period* while it does affect the *area*, the conclusion which Newton states in one of his corollaries is unavoidable, viz: "the periodic times in ellipses are the same as in circles whose diameters are equal to the major axis of the ellipses." The major axis of the ellipse alone will not enable us to determine its area, which may be anything from 0 to the circle whose radius is semi-major axis; hence the expression $T^2 \propto A^3$ forbids a relation between the *periodic time* and the *area* or *velocity*.

Newton concedes in his corollaries that all orbits having the same major axis and the same central force will be described in the same time. He asks us to believe as has been stated before that in the case of the highly attenuated ellipse in which the centre of

force is near the extremity of the axis, the body will turn abruptly about the centre of force and return to the other extremity almost upon the axis, turn short again and thus continue.

Since it is impossible to submit such conclusions to a practical test, though they may be repugnant to common sense, so long as they bear the seal of demonstration and baffle disproof, what better can be done than to accept?

Notice Newton's suspicious evasion. The limits of the ellipse are, on one side, the circle whose diameter is the major axis, and on the other the major axis itself. It is but just that we demand of a law prevailing between limits, that the limits are *included.* On the side of the circle Newton says the limit is included but the other limit he says is *excluded.* If this limit were not excluded, Newton's theory of orbital motion would end in this: that when the ellipse is so narrow that the focus occupies the *extremity* of the axis, a body moving from the other end of the axis in a straight line, from rest, toward the centre of force in the other focus, the moving body would, if unobstructed by actual contact, proceed to the centre of the central force and then return to its starting place at the other end of the axis. It is solely to evade this culmination of absurdity that Newton precludes the case of a body falling from a state of rest. I say culmination of absurdity because the near approach to the limit is but little less absurd. This failure on the part of Newton to harmonize the laws of rectilinear with those of curvilinear motion was the first inducement to the present investigations.

According to the new doctrine of orbits, the theory of *precipitation* supplies the consistencies so manifestly wanting in the elliptical.

So far as the author of this work is able to discern, all the false conclusions to which Newton attained owe their incorrectness, directly, or indirectly, to the two mistakes noted before, viz: his failure to recognize the constancy of a chord passing through two given points and his violation of an axiom by asserting the equality of a whole and one of its parts. Subsequent to these mistakes his demonstrations are so lucid and his conclusions so just that one almost feels sorry that so masterly an effort should be rendered so wholly worthless.

Prop. 16, with its nine corollaries discussing the relations of velocity and parameter as adapted to conic sections may, by a

proper change of terms, be harmonized with the new theory. Prop. 17 would seem to settle all doubts when it determines the orbit of any body projected with any rate of motion in any direction, subject to a given central force. The conclusions of these latter propositions like many of the others are false only by inheritance from the fundamental errors.

Criticism on Newton might properly be extended to numerous suppositions which he introduces at the conclusions of his demonstrations, but since his corollaries and observations must share the fate of the general doctrine, and since also it is the purpose of the present effort more especially to establish the new than to expose the old, those minor features will not be detailed upon at present.

Though not strictly pertaining to orbits, we can not refrain from observing that Newton has carried his error along in the treatment of rectilinear motion, as may be seen by his handling of Prop 32. "To find the time of motion and velocity acquired when a body falls through a given space from rest under the action of a force which varies inversely as the square of the distance."

It is by assuming that the ellipse satisfies the law of gravitation (which is the mistake), that he derives the time of describing an elliptical arc independent of the minor axis, from which he could not otherwise than conclude that a body will describe all ellipses of the same major axis in the same time, in which is included rectilinear motion in the major axis as an ultimate. Even if the elliptical theory of curvilinear motion were correct, serious exception might be taken to Newton's solution of this problem. That he appreciates the existence of a dilemma may be inferred from the notes of explanation, which are, by the way no explanation at all, but a clearer statement of the difficulty. He will have the centre of force in the focus of the ellipse up to its ultimate narrowness, stating that the revolving body turns suddenly about the central force to complete its narrow orbit. He begs, however, without a just claim to allowance, that in the case of rectilinear motion, the body be excused from complying with the law which applies to the ultimate ellipse; that in this case the body instead of reversing its motion in the highest state as it arrives at the centre, be permitted to pass the centre of force a distance equal to the line of approach.

The new theory of orbits finds no obstacle in the way of harmonizing rectilinear with orbital motion. In view of the impossibility

of harmonizing Newton's theories of curvilinear and rectilinear motion, it is a question as to which is the greater wonder,—that Newton should demand, or that the talent of ages should yield, assent to so evident a self-contradiction. Whenever Newton performs a demonstration upon the ellipse with a force varying as the distance tending to the centre, his conclusions are satisfactory, and we reassert that this is the only law and condition by which the ellipse may represent orbital motion subject to a single force. Since, however, such a law does manifestly not rule the universe, the ellipse, as an orbit, is ignored altogether. As for the parabola and hyperbola, they have no higher claim than the ellipse and can be quite as easily dispensed with. The new doctrine of orbits will by means of the circle alone account for all the movements noted within the limits of the solar system, and if not of the entire physical universe, at least as far as the law of inverse square prevails.

Another mongrel conception of Newton's by which he evades precipitation is found in his observations on the famous Prop. 7, in which he discusses the case of motion in which the central force is external to the curve.

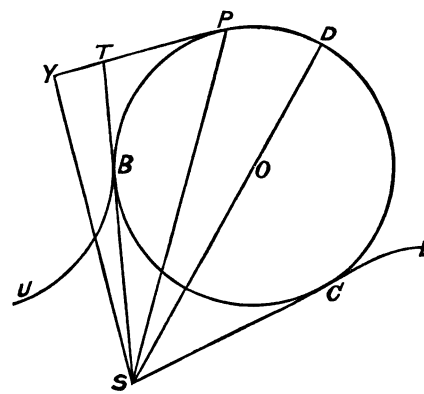


Fig. 33.

In the treatment of this case Newton assumes that the force tending to S is attractive for the arc $C\,D\,B$ and repulsive for the arc $B\,C$ included between the tangents to the curve drawn from S. He declares the possibility of moving in the several arcs under the assumed forces, but admits the impossibility of passing from one arc to the other. He says that when the body moving from P arrives at B it will cross the tangent at B, describe $B\,U$, continuing the similar circle till it arrives again at B where it will pass to the opposite side of the tangent to describe the first circle. Now Newton says it is impossible for the body to continue in the same circle at B. At C, however, where exactly the same obstacle is in the way he asserts the possibility of description. In order to be con-

sistent, he should say at C the body will cross the tangent and describe the arc CE &c.

Pursuing this mode of treatment the path of the moving body will be quite a novel figure indeed, never except by rare accident repeating its former course.

Newton tries to prove that a finite velocity perpendicular to SB remains up to B, which causes the body to pass to the opposite side of the tangent, which he does by saying that the component of velocity at P perpendicular to SB when P is very near B, is $\frac{h}{SY}, \frac{SY}{ST} = \frac{h}{ST} = \frac{h}{SB}$.

This value $\frac{h}{SB}$ he considers finite even at B, but let us see; as the point P approaches B the tangential force inclines more and more in the direction of the central force until, at B, they both tend in the *same* direction. Now h represents an area, that is, the parallelogram of the tangential and central forces. When the forces tend in the same direction as at B there is no parallelogram, since the forces tend in the same straight line, hence, at B the value of h is 0 $\therefore \frac{h}{SB}$ is 0, $\therefore$ there is no force at B perpendicular to SB.

Although the possibility of curvilinear motion with a force external is wholly denied, granting in this instance that the body may describe the arc PB, it is claimed that from the moment that the two forces tend in the same direction SB, all the future motion of the moving member, whether the force from S be attractive or repulsive, is confined to the straight line ST, resulting in precipitation or departure in a right line. This result is evident when we remember that curvilinear motion is the result of warring forces —centrifugal tending in a straight line irrespective of the position of the central force, and centripetal tending constantly to the center. So long as tangential force is strong enough, there will be an angle between its direction and the radius vector, and consequently an area, but when tangential force is so weak that its direction comes in line with the central force, or the radius vector, it is evident that the mastery is with the central force, that both forces tend thenceforth in the same direction, and the result is rectilinear precipitation. Just at the point where both forces tend

in the same direction, Newton asserts that the central force becoming repulsive the body passes to the opposite side of the tangent, there describing a similar circle, thus passing alternately from branch to branch of the complex orbit.

He does not indicate what would be the future of the moving body if the force continued attractive at B: The reason that Newton demonstrates himself away from precipitation is, no doubt, because it endangers his entire family of orbits.

Admitting the possibility of precipitation, he could not defend the narrow ellipse which he could ill afford to lose.

It may be very easily shown that the possibility of curvilinear motion tending to an external force is a mathematical absurdity. It is proven by Newton's 1st and 2nd propositions that the description of equal areas is an essential feature of orbital motion. We have shown that whatever the value of h in any other part of the assumed orbit, its value at B is 0, hence it is not a constant value. This circumstance alone denies the possibility of the assumed movement or the correctness of fundamental propositions.

It may reasonably be asked—if Newton's demonstrations and Kepler's announcements corncerning the form of orbits are so utterly wrong, how is it that the purposes of the astronomer are so well served?

We reply, that the orbits with which the astronomer has most to do, are so near the limit where the ellipse and the circle coincide that it does not materially interfere with observations and calculations to mistake one for the other.

SECTION VI.

The Anomaly Explained by the Circular Theory.

The facility with which the anomaly can be explained by the true doctrine of orbits, is one of the strong features in its favor. There is no need of recourse to another curve, as is done in treating the anomaly from an elliptical orbit. Loomis takes advantage of a circle, whose radius is equal to the square root of the product of the semi-axes, and Newton uses the circle, whose diameter is the major axis of the ellipse. The peculiarity of this combination used by Newton is that the circle, which he appropriates simply for assistance, is the *real* path of the body moving under his given conditions, from perihelion to aphelion. No wonder that the conclusions derived from this unconscious use of the *true* orbit affords results so satisfactory to the observer. In this particular there will be no material diversity of results, only they will be derived from the orbit directly and in an easier manner.

Suppose APB, the orbit of a body moving about a central force at S, A and B, being the apsides, and P the position of the body at any time.

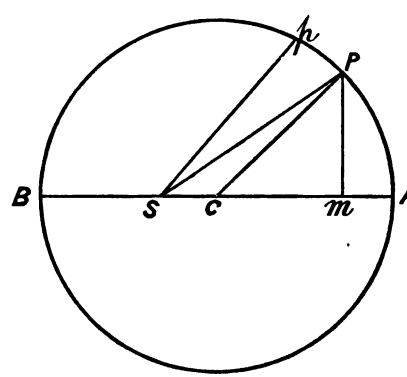


Fig. 34.

Suppose also p an imaginary body independent of the central force, following in the given orbit with a velocity such that it will complete a revolution in the same time as P, by describing a uniform *angular* motion about S, both bodies being at A at the same time. Then, by defining the anomaly as the angular distance of a

planet from an apse, the angle ASP represents the *true* anomaly, ASp the *mean*, and ACP the *eccentric* anomaly. PSp is the *equation of the centre.*

To find the time of an apse to any point of an orbit.

Time AP : periodic time :: area $ASP : \pi AC^2$.

Area $ASP =$ sector $ACP +$ triangle CSP.

Area $ASP = \frac{1}{2} AC, AP + \frac{1}{2} CS, Pm$.

Area $ASP = \frac{1}{2} \overline{AC}^2 (u + e \sin. u)$, if u be the circular measure of the angle ACP to a radius of unity and e be the eccentricity of the orbit.

$\therefore$ time $AP : \frac{2\pi AC^2}{f^{\frac{1}{2}}}$ (Th. IX) :: $u + e \sin. u : 2\pi$, hence the time from the apse A to the point $P = \frac{AC^2}{f^{\frac{1}{2}}} (u + e \sin. u)$.

Similarly if the body were moving from the nearer apse B in the opposite direction, it may be shown that the time in $AP = \frac{AC^2}{f^{\frac{1}{2}}} (u - e \sin. u)$.

In the expression which shows the time in $AP = \frac{AC^2}{f^{\frac{1}{2}}} (u + e \sin. u)$, it will be observed that if all the terms but AC are constant, the time in $AP \propto R^2$, R being the radius of the orbit. If all the terms but f are constant, time $\propto \frac{1}{f^{\frac{1}{2}}}$ *i. e.* inversely as the square root of the force, *e. g.* if the central force were 4 times as great the time in describing an arc would be $\frac{1}{2}$ as great. If all the terms but u or e were constant, the time in AP would *depend* on the variable, but would not vary directly, since u and sin. u do not vary directly. If, however, e were 0 the time would depend on u directly, velocity being uniform.

An important feature belonging alone to the new orbit is that the *eccentric* anomaly and the *actual* velocity of the planet are alike represented by the progress of the radius.

Since the mean angular velocity in an orbit is 2π divided by the periodic time, we find from the proportion regarding *time in AP* as unity.

$$1 : \frac{2\pi AC^2}{f^{\frac{1}{2}}} :: u + e \sin. u : 2\pi.$$

$$u + e \sin. u = 2\pi \div \frac{2\pi AC^2}{f^{\frac{1}{2}}}.$$

Whence, if m is the mean anomaly corresponding to the eccentric anomaly u, we have $m = u + e \sin. u$.

To find the *true* anomaly $A\,S\,P$.

Let v be the true anomaly, $A\,S\,P$,

w the angle, $C\,P\,S$.

Let Fig. 33 represent the orbit of the earth, of which $C\,P$ is the radius R and $C\,S$ the eccentricity, e.

By trig. $R + e : R - e :: \tan. \frac{1}{2} u : \tan. \frac{1}{2} (v - w)$.

If e be unity and u, be 60°, R being 60 we have 61 : 59 :: .5773 : .5747,

:: tan. 30° : tan. 29° 53′.

Whence $v = 59° \; 53'$

$w = 7'$.

$m = u + e \sin. u = 60° \; 50'$

∵ the equation of the center $= m - v = 60° \; 50' - 59° \; 53' = 57'$.

Assuming the eccentric anomaly $u = 90°$, we derive the following results:

61 : 59 :: 1 : .9672,

:: tan. 45° : tan. 44° 2′

∵ $v = 89° \; 2'$ and $w = 58'$

$m = u + e \sin. u = 90° \; 57'$, and the equation of the center is 1° 55′.

In the instances of the anomaly and the treatment of cometary orbits we meet with unexpected agreements between the old and the new doctrines.

The cause of this agreement is the accidental appropriation of principles to the elliptical theory which belong properly alone to the circular.

This treatment of the anomaly will render the preparation of an ephemeris a task of comparative ease. Many of the laborious manipulations heretofore in use will be consigned, together with Newton's demonstrations and Hamilton's ingenious Hodograph, to the cabinet of scientific curiosity.

The equation of the centre for a cometary orbit may be quite as readily obtained by the method just given.

The radius to Enck's orbit is 212,000,000, its eccentricity 179,000,000. If we let 1 represent the radius, .84 + represents e.

To find the equation of the centre when the comet has advanced $\frac{1}{4}$ of a revolution from aphelion, when $u = 90°$, the general proportion.

$R + e : R - e ::$ tan. $\frac{1}{2} u$: tan. $\frac{1}{2} (v - w)$, becomes 1 .84 : .16 :: 1 : .087,

:: tan. 45° : tan. 4° 58′.

Whence $v = 49° 58'$.

$w = 40° 2'$.

$m = u + e$ sin. u $= 130° 2'$.

$m - v$ or the equation of the center $P S p = 80° 4'$.

Or suppose the comet to move from the nearer apse B, to the same point P, then $B S P$ or $v = 180° - 49° 58'$ $= 130° 2'$. The value of m, reckoning from the nearer apse, $= u - e$ sin. $u = 90° - 40° 2' = 49° 2'$, and the equation of the centre, the center is $v - m = 80° 4'$, the same as above.

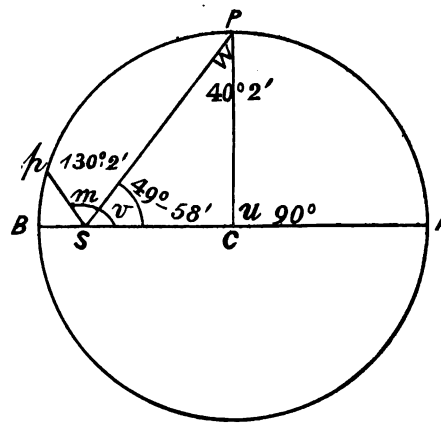


Fig. 35.

Find the equation of the centre for Halley's comet having advanced 60°* from perihelion.

The eccentricity of Halley's orbit is about 30 times perihelion distance, whence the general proportion $R + e : R - e ::$ tan. $\frac{1}{2} u$: tan. $\frac{1}{2} (v - w)$, becomes 61 : 1 :: tan. 60° : tan. 2°, 37′.

hence $v = 117° 23'$ and $w = 57° 23'$.

$m = u - e$ sin. $u = 3° 4'$.

$\therefore$ equation of the center $= v - m = 114° 18'$.

Find the equation of the centre for Halley's comet having advanced 140° eccentric anomaly from perihelion or 40° from aphelion.

1st. From aphelion:

$R + e : R - e ::$ tan. $\frac{1}{2} u$: tan. $\frac{1}{2} (v - w)$ becomes 61 : 1 :: tan. 20° : tan. 20′.

*Eccentric anomaly.

Whence $v = 20° 20'$, $w = 19° 40'$.

$m = u + e \sin. u = 40° + 38° 28' = 78° 28'$, and equation of centre $= m - v = 58° 8'$.

2nd. From perihelion:

$61 : 1 :: \tan. 20° : \tan. 20'$.

Whence $v = 159° 40'$, $w = 19° 40'$.

$m = u - e \sin. u - 140° - 38° 28' = 101° 32'$, and equation centre $= m - v = 58° 8'$, agreeably the same result either way.

The foregoing results obtained by application of the general proportion belonging to the anomaly are only approximate, small fractions being neglected. It will now be an easy task to prepare an ephemeris for a planet or a comet of known period. The object of the present work, however, being most particularly to establish the true doctrine of orbits, the preparation of detailed tables will be reserved for the future.

SECTION VII.

Orbital Motion About two Forces.

The possibility of describing an elliptical orbit about a single central force is stoutly denied. Under the influence of two equal forces, one in each focus, it is possible that an ellipse may be described. The case is special and disallows an eccentricity much greater than half the major axis.

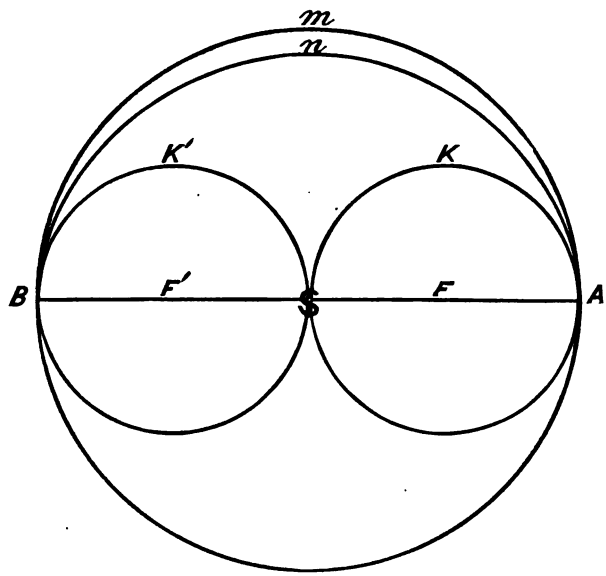


Fig. 36.

Suppose Fig. 35 a body moving in the non-eccentric orbit, *A m B* about the central force *S*. While the body is at *A*, place ½ of *S* at *F*, then will the body move with unchanged velocity about *F* in ½ of the time it moved about *S*, assuming that the other half of *S* is removed.

For, periodic time of an orbit $= \frac{\pi A C^{\frac{3}{2}}}{F^{\frac{1}{2}}}$, $A C$ being radius of the orbit and F the central force. $A C$ and F both being taken $\frac{1}{2}$ as in the Fig. $P = \frac{\left(\frac{1}{2}\right)^{\frac{3}{2}} \pi A C^{\frac{3}{2}}}{\left(\frac{1}{2}\right)^{\frac{1}{2}} F^{\frac{1}{2}}} = \frac{\frac{1}{2} \pi A C^{\frac{3}{2}}}{F^{\frac{1}{2}}}$, that is, the period is $\frac{1}{2}$ as long.

The velocity is unchanged since the circumference $A K S = \frac{1}{2} A m B$. In order that the body at A may describe $A m B$ about F, the velocity at A must be increased.

If, at the same time that the velocity of the body at A is increased, the other half of S be placed in F', the tendency of F' will be to draw the body within, $A m B$ causing it to move in $A n B$, a curve, which by special conditions will satisfy the equation of the ellipse. Most generally, however, the figure $A n B$ is pseudoelliptical. The compound orbit into which this case may be developed will be considered again.

In the course of these demonstrations certain properties have been noticed which may be pardoned as digressions.

$x^2 + y^2 = r^2$, when r is constant, is the general equation of the circle; but, when r is variable, the same equation may represent the ellipse referred to a focus as origin of co-ordinates.

Select the right-hand focus as origin, and let r be the vector from that focus, while r' is vector from the other focus, e being the eccentricity.

Then $x^2 + y^2 = r^2$.

$(x + 2e)^2 + y^2 = r'^2$.

$(x + 2e)^2 - x^2 = r'^2 - r^2$.

$(x + 2e) + x : r' + r :: r' - r : (x + 2e) - x$.

$2x + 2e : r' + r :: r' - r : 2e$.

$x + e : r' + r :: r' - r : 4e$; that is, the abscissa + the eccentricity is to the sum of the vectors as the difference of the vectors is to four times the eccentricity.

From $x^2 + y^2 = r^2$, as applied to the ellipse, the limiting values of r may be found by making $y = 0$, when $x^2 = r^2$, $x = \pm r$, or $r = \pm x$. Here it might appear that r has two equal values of opposite sign, but the fact is other. For, if A be the semi-major

axis, the positive limit of x is $A - e$, while the negative limit is $A + e$, hence the limiting values of r are $A - e$ and $A + e$.

Supposing $y = 0$ in $(x + 2e)^2 + y^2 = r'^2$, we obtain $r' = \pm (x + 2e)$ which, by positive interpretation, is $r' = x + 2e$, and by negative $r' = -(-x + 2e) = x - 2e$. Or, positive $r' = A + e$, and negative $r' = A - e$, since $-x = A + e$, all of which agrees with the alternating values of r and r' whose sum is constantly $2A$.

When $x = 0$, in $x^2 + y^2 = r^2$, $r = \pm y$, which, being interpreted in harmony with the symmetry of the curve, gives two equal and opposite values. In order to interpret the limiting values of functions, it is important to keep in view the properties of the curve represented by the given equation.

For example, there is quite a difference in the expression $x = \pm r$, whether the equation refers to a circle or to an ellipse.

THEOREM XVI.

The equation of the ellipse, referred to a focus and the major axis, is

$$x = \frac{B^2 - Ar}{e}.$$

$2x + 2e : r' + r :: r' - r : 2e$, but $r' + r = 2A$, and $r' - r = 2A - 2r$.

Hence, $2x + 2e : 2A :: 2A - 2r : 2e$.

Or, $x + e : A :: A - r : r$.

$$x = \frac{A^2 - Ar}{e} - e = \frac{(A^2 - e^2) - Ar}{e} = \frac{B^2 - Ar}{e}.$$

In this result, A and e are the constants and x and r the variables. The expression is independent of ordinate and vector of the remote focus, and in $x = \frac{A^2 - Ar}{e} - e$ of the minor axis, but these terms are so intimately related to those given in the equation that their values are readily found. This result is obtained with origin of coordinates at right-hand focus. By proper change of signs it applies equally to the other focus. x is positive when $\frac{A^2 - Ar}{e}$ is positive and greater than e, and its positive value is greatest when r is least, which is when $y = 0$.

x is negative when $\frac{A^2 - Ar}{e}$ is negative or positive, and less than e. The greatest negative value of x is when r is greatest. $x = 0$ when $\frac{A^2 - Ar}{e}$ is positive and equals e. When $r = A$, $x = -e$, whence by substitution the proportion,

$x + e : A :: A - r : e$ becomes

$o : A :: o : e$, that is, o bears the same relation to all quantities.

The value of the semi-minor axis is $\sqrt{A^2 - e^2}$, of r' the value is $2A - r$, and of any ordinate $\sqrt{r^2 - x^2}$ hence, the terms found in the equation.

$x = \frac{A^2 - Ar}{e} - e$ supply all the wants of the ellipse. As e approaches A in value the ellipse approaches a straight line represented by the major axis, which it becomes in the limit. Then $x + e : A :: A - r : e$ becomes $x + A : A :: A - r : A \therefore x = -r$. But in this case $r = o$ or $2A$, hence $x = o$ or $-2A$.

As e approaches o the ellipse approaches the circle upon its major axis which it becomes in the limit.

For $(x + 2e)^2 + y^2 = r'^2$ which, when $e = o$, becomes $x^2 + y^2 = r'^2$; but $x^2 + y^2 = r,^2$ hence $r' = r = A$, a constant; hence the equation $x^2 + y^2 = r^2$ or r'^2 is the equation of a circle.

$r = A - \frac{ex + e^2}{A}$, derived from the preceding proposition, is a convenient formula to derive the value of the radius vector to a given point of the ellipse, when the eccentricity of the curve and the abscissa of the point are given.

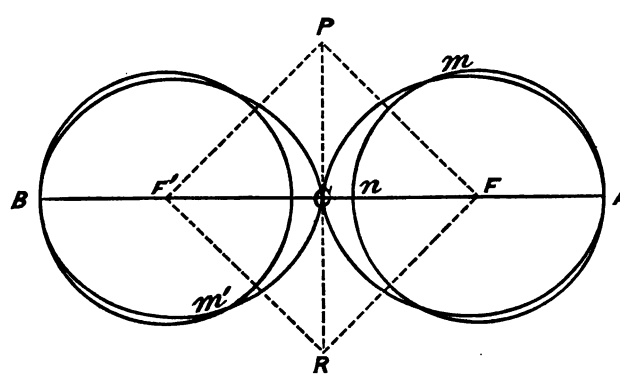


Fig. 37.

Fig. 37 illustrates the conditions of a compound orbit. Suppose F and F' the positions of two equal forces, and A the position from which a body is projected at right angle with a given velocity. The development of the circular theory shows that there is a degree of motion, by which, if undisturbed, the body moving from A will describe the circle $A\,m\,n$. The arrangement of the forces F and F' may be such that F' will cause the body moving from A to describe the curve $A\,m\,C$. So long as the velocity of the body is not quite sufficient to reach the point C the body will describe a modified orbit about F. If, however, the adjustment of forces will cause the body to reach exactly to C, the body will cross to the opposite side of $P\,R$ and pursue the similar course, $C\,m'\,B$ about F'. The body will arrive at B with the same momentum as it had at A. The conditions being the same at B as at A the body will return to A in the curve indicated. Taking into account the totals of disturbance, an endless variety of complicated movements about various centres of force are among the possibilities of the law of gravitation.

If the velocity imparted to the moving body be greater than that needed to carry to C (midway between F and F'), the body will cross the line $F\,F'$ at a point beyond C. Following the various degrees of acceleration it will be observed, that moderate increase will result in a permanent orbit about F' in the direction of dial movement, modified by disturbance from F. A farther increase of velocity in the body as it moves from A, results in precipitation upon F'. Still higher acceleration at A results in a modified orbit about F' in the direction opposite to dial movement, while still larger increase to the momentum at A results in a simple orbit about both F and F', in form either elliptical or pseudo-elliptical.

The various degrees of momentum with which the body moving from A passes beyond the sphere of influence of F, resulting in *direct*, *indirect* and *variously* eccentric orbits about F', afford an excellent illustration of cometary behavior, assuming that the origin of comets is foreign to the solar system.

Another interesting case may be illustrated by Fig. 37. Suppose a body from rest at C, midway between F and F', be propelled in the direction $C\,R$ at right angles with $F\,F'$. Since the forces F and F' are equal and opposite, the motion imparted to the body at C will continue in the right line $C\,R$, until the body is brought

to rest by the retarding effect of the forces. The distance CR depends upon the momentum of the moving body, the intensity of the forces, and the distance CF.

After coming to rest at R the body will return in the same path to C, having the same momentum with which it left C, tending in the opposite direction PC. The behaviour of the moving body so long as undisturbed, will now be vibration between the points P and R.

If at the point R, or any place in CR, a new additional impulse acts upon the moving body in a plane at right angle with the axis AB, and also the line CR, the resultant will be an orbit in the new plane, every point of which is equidistant from F and F'. The force tending from R to C, it must be observed, however, does not vary according to the law of gravitation; hence the last-named orbit will not comply with the laws of actual planetary motion. The force tending to C is one that varies between two zeros, being 0 at C, and again at an infinite distance in CR. The point of maximum intensity in the line CR is determinate.

Problem, to find the point in CR, at which the tendency to C is maximum:

Regard the distance CF' and the force F' at C as unity, and let $x =$ the distance in CR, at which the force tending to C is maximum.

Assuming that R is the required point we find from the triangle, $F'CR$.

$FR = \sqrt{1 + x^2}$, whence the force of F' at R, since it varies inversely as the square of the distance, equals $\frac{1}{1 + x^2}$.

Resolving the force $\frac{1}{1 + x^2}$ into its components CF' and CR, $\frac{1}{1 + x} \times \frac{1}{1 + x^2} = \frac{1}{1 + x + x^2 + x^3}$ represents the value of the force tending to F', and $\frac{x}{1 + x + x^2 + x^3}$ the part tending to C. It may readily be shown that $\frac{x}{1 + x + x^2 + x^3}$ is maximum when $x = 1$, and that $\frac{1}{1 + x + x^2 + x^3}$ is maximum when $x = 0$; hence the force tending to F' is greatest when the

body is at C, and the force tending to C is greatets when the distance of the body from C is equal to the distance of C from F'.

Another condition of the problem of three bodies is illustrated in Fig. 38.

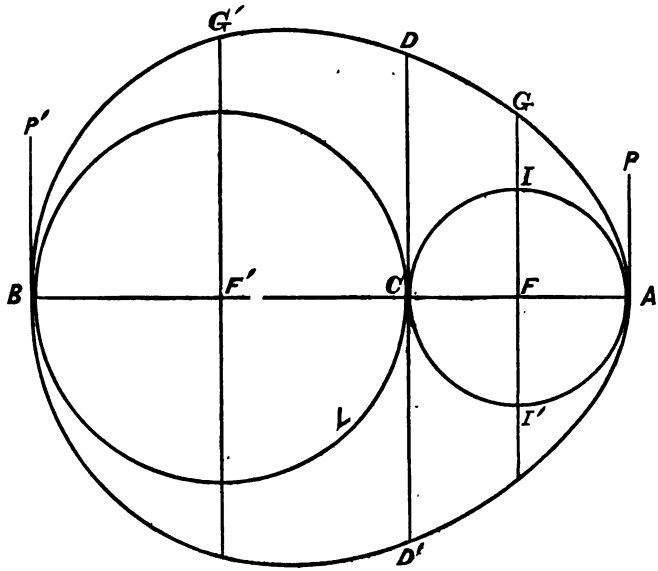


Fig. 38.

Let F and F' be the positions of two *unequal* forces, so related that their intensities are equal at C. If a body at A be propelled in the direction $A\ P$, the path will vary in accordance with laws established.

Falling from a state of rest at A the body precipitates upon F in the straight line $A\ F$. Various degrees of feeble impulse in the direction $A\ P$ result in various arcs of precipitation. Increased centrifugal force satisfies various eccentric orbits about F, of which A is aphelion.

When centrifugal force, together with the perturbation from F', is sufficient to reach C the body will cross to the opposite side of the tangent line $D\ D'$, and perform the orbit $C\ L\ B$ with the radius $C\ F'$. At C the body will cross the tangent line again and make its way to A in the semicircle $C\ I'\ A$.

By increased velocity various stages, similar to those in Fig. 37,

will be passed until finally the body will describe the orbit $A\,D\,B\,D'$, which we designate as the egg-shaped oval, some of whose properties it is proposed to develop.

It will be observed that the construction of the egg-shaped orbit depends upon the ratio of the forces F and F'.

Fig. 38 assumes $F' = 4\,F$, whence $F'\,C = 2\,F\,C$.

The discussion will be conducted, however, in general terms.

With $C\,D$ and $C\,A$ as axes and C the origin of co-ordinates, $C\,F = a$, $C\,F' = K\,a$, r and r' the radius vectors from F and F' respectively, we derive, x and y being the general co-ordinates:

$$(x - a)^2 + y^2 = r^2.$$
$$(x + Ka)^2 + y^2 = r'^2.$$
$$x^2 - 2\,ax + a^2 + y^2 = r^2.$$
$$x^2 + 2\,aKx + a^2\,K^2 + y^2 = r'^2$$
$$a^2\,K^2 + 2\,aKx + 2\,ax - a^2 = r'^2 - r^2.$$
$$(aK + a)\,(aK - a + 2\,x) = r'^2 - r^2, \text{ from which}$$

THEOREM XVII.

The difference of the squares, of the radius vectors, of the egg-shaped orbit is equal to the sum of the focal distances, multiplied by their difference plus twice the abscissa of the point.

From the general equation we find $x = \dfrac{r'^2 - r^2 - (K^2\,a^2 - a^2)}{2\,(K\,a + a)}$, that is, the value of the abscissa to any point in the egg-shaped oval, equals the difference of the squares of the radius vectors, minus the difference of the squares of the focal distances, divided by twice the sum of the focal distances.

Observations on Theorem XVII.

1. In the equation $r'^2 - r^2 = (K\,a + a)\,(K\,a - a + 2\,x)$, when $x = 0$, $r'^2 - r^2 = K^2\,a^2 - a^2$; that is, the difference of the squares of the radius vectors to the point at which the curve cuts the axis of ordinates is equal to the difference of the squares of the focal distances.

When $K\,a = a$, $r' = r$, in which case the curve is symmetrical with reference to the axis of ordinates.

2. In $(x - a)^2 + y^2 = r^2$, and $(x + a\,K)^2 + y^2 = r'^2$, when $x = -2\,a\,K$, $y = o$.

$(-2aK-a)^2 = (2aK+a)^2 \therefore, r = 2aK + a$, the distance from F, at which the curve cuts the axis of abscissa. The distance at which the curve cuts the abscissa from F is similarly found to be aK.

In the same equations if $x = 2a$, $y = 0$.

Whence $r = a$ and $r' = 2a + aK$.

3. When x in the same equations $= a$,

$$y^2 = r^2 = FG^2.$$

$$(a + Ka)^2 + y^2 = r'^2 = F'G^2,$$

verified by $F'G^2 = FF'^2 + FG^2$.

The same results from $x = -aK$.

In Fig. 38 the velocity at B is the same as the velocity at A. The difference of the curvatures at A and B is owing partly to the same causes as the difference in the included circles, and partly to the difference of the collective forces at those points.

For example, if $F' = 4F$ and the effect of F at the distance $FA = 1$, the united effect of F' and F at $A = 1\frac{1}{4}$ while at B, their united effect which is the measure of deflecting force at those points is $1\frac{1}{25}$.

The foregoing results pertaining to Fig. 38, belong only to the relations between focal distance and abscissa.

The several equations are satisfied at any value for the ordinate. The value of y depends in this problem of forces upon the relative intensities of F and F', upon the velocity of the moving body and upon the retarding and deflecting factors of the central forces. The development of this curve involves tedious operations conducted in sympathy with the following problem.

PROBLEM.

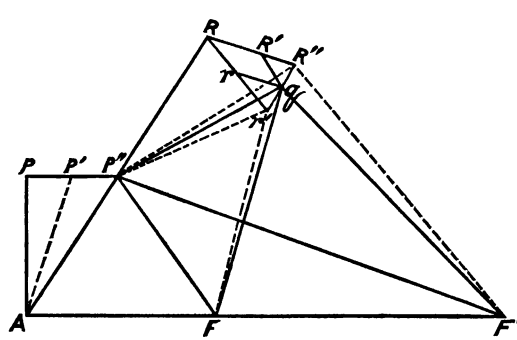


Fig. 39.

A body is moving about two central forces subject to the law of gravitation, find the variations in the areas described by the radius vectors to the forces.

Let F and F' be the positions of two central forces of known magnitude, A in the line passing through both forces, the position of a body moving with a given velocity in the direction $A\,P$, at right angle to $A\,F'$. If the force F, acting by instantaneous impulse at A, is sufficient to deflect to P', while the body moves from A to P, and the force F' is able to cause the farther deflection $P'\,P''$, then, clearly, P'' will be the position of the moving body at the end of the given interval of time.

It will be observed that, in this instance, the areas described by the vectors $F\,P''$ and $F'\,P''$ are the same as if either or neither of the forces acted upon the moving body, for $A\,P\,F = A\,P'\,F = A\,P''\,F$ and $A\,P\,F' = A\,P'\,F' = A\,P''\,F'$. Hence, also the *sum* of the areas described by the vectors in the given time, is the same as if either or neither of the forces disturbed the motion of the moving body.

In another equal interval of time the body would, if free to move, pass to R in the direction $A\,P''$, $P''\,R$ being equal to $A\,P''$.

If $R\,r$ is the effect of deflection from F and $R\,R'$ the effect of deflection from F', then, $R\,r$ and $R\,R'$ being parallel to $P''\,F$ and $P''\,F'$ respectively, and $r\,q$ being equal and parallel to $R\,R'$, the true position of the body at the end of the second interval is q, $R\,q$ being the resultant of $R\,r$ and $R\,R'$.

If the effect of both forces at P'' had tended in the direction $P''\,F$ the body at the end of the second interval would be at r', $r\,r'$ being equal to $R\,R'$. In this case the area by the vector to F would be equal to that described in the first interval. That is $P''\,r'\,F = A\,P''\,F$. By reason of the action of F' the area by the vector to F is increased by the sum of the triangles $P''\,r'\,q$ and $F\,r'\,q$.

Similarly, if at P'' both forces tended with their given intensities in the direction $P''\,F'$, the body would be at R'' at the end of the second interval, $R'\,R''$ being equal to $R\,r$. $P\,R''\,F'$ would then be the area described by the vector to F'. Since q is the position of the body at the end of the second interval, it is plain that the area by the vector to F' is diminished by the sum of the

triangles $P'' R'' q$ and $F' R'' q$. Equal areas in equal times about F demands the body somewhere in $R r'$ produced, about F' in $R R''$.

If K represents the sum of the areas described by the vectors, in the first interval of time, the sum of the areas during the second interval is

$$K + (P'' q r' + r' q F) - (P'' q R'' + R'' q F').$$

As the interval of time is indefinitely diminished, the sum of the areas $P'' q r'$ and $r' q F$ will differ by less than any assignable value from $P'' F \times q r'$, and the sum of $P'' q R''$ and $R'' q F'$ by less than any assignable value from $P'' F' \times q R''$.

This limitation reduces polygonal to curvilinear motion about two central forces; hence, if K represents the sum of the areas described about two central forces in an infinitely small interval of time, the sum of the areas for the next interval of time is

$$K + (P'' F \times r' q) - (P'' F' \times q R').$$

In the similar isosceles triangles $r r' q$ and $R' R'' q$, $R' q$ or $R' R''$ expresses the intensity of F at P'', and $r q$ or $r r'$ the intensity of F' at P''.

Since $R' R'' : r r' :: R'' q : r q$, the force of F at P'' : force F' at P'' :: $R'' q : r q$.

The general expression for the sum of the areas by the vectors to two central forces is $K + (P'' F \times r' q) - (P'' F' \times q R')$.

It will be observed, that the value of this expression depends upon the *intensity* and *distance* of the forces, and also upon the angle which the vectors make with the line passing through both forces.

It will be observed, that in passing from quadrature to an apse the order of computation must be reversed—that the areas which are added to K in passing from apse to quadrature, must be subtracted in passing from quadrature to apse.

In the case of the pseudo-ellipse with two equal central forces the sum of the areas decreases from the apse to quadrature. At quadrature it is unaffected, and increases from quadrature to the apse.

The velocity is maximum at the apse, for, if it were uniform the area at the apse would be minimum.

When the central forces are brought together, the expression $K + (P'' F + r' q) - P'' F' \times q R'')$ becomes constant, equal to K; for, when the forces tend to the same point $r' q$ and $q R''$ are

both zero. This condition agrees with motion about a single central force, describing equal areas in equal times.

By the aid of these principles, if it were needed, the orbit of a body moving about two or more central forces might be determined with exactness; but, since there are no known movements among heavenly bodies demanding the solution, this supposable case is left to its merits. The excuse for presenting this case is, that since culture is one of the chief ends in the development of science, the presentation of the *possible* may reasonably be associated with the actual.

Of the orbital motions thus far considered, the circle alone admits of general application. Given a central force and a moving body, the discussions pertaining to the circle satisfy all the variations of distance, intensity and velocity. The pseudo-ellipse and the egg-shaped oval on the other hand are only special cases.

Given two fixed central forces and a moving body, there is only *one* fixed rate of motion that will satisfy the curves. They are quite as special as the non-eccentric, circular orbit.

If the velocity in an ellipse be increased or diminished, the sum of the radius vectors will be increased or diminished; hence, the curve will no longer satisfy the laws of elliptic construction. A strictly general treatment of motion about two central forces is possible, but so long as such conclusions cannot be utilized they will fail to command earnest consideration. The solution of the problem Fig. 39 is general, applying to all possible variations. It applies as well to the perturbing effects of planets upon each other as to the given case.

In the discussion on Fig. 39, it is presumed possible that two central forces may exist in *fixed* positions. It being necessary that those forces revolve about their centre of gravity, the motion of the three bodies is rendered far more complicated than in the case supposed. The same remark applies also to planetary motion. Their central force, the sun, is regarded as fixed, by which are derived the properties of orbits *relative* to the sun. Since, however, the sun is also moving, it is manifest that the *absolute* movement of planets is quite different from the relative, much more complicated if not, indeed, "past finding out." The difficulty before us in deciphering the *absolute*, should not and does not divest the knowledge of the *relative* of its value; for, on this plane all the departments of science are equally impotent.

SECTION VIII.

Secondary Orbits and General Properties of the Cycloid —The Tidal Wave and a Theory of Common Waves.

The sun moving about some undetermined centre, the motions of the planets about that centre are similar to the motion of the moon with reference to the sun.

Preparatory to a farther consideration of secondary movement the following discussion on the cycloid is offered.

GENERALIZATION OF THE EQUATION OF THE CYCLOID.

The equation of the cycloid as developed is $x =$ arc whose ver. sin. $= y, - \sqrt{2ry - y^2}$, which is the curve traced by the generating point upon the circumference of the rolling circle.

The generating point may, however, occupy any position upon a radius of the rolling circle, from circumference to the centre, by which an endless variety of curves may be described. The curve described by any other point than that upon the circumference of the rolling circle is called the *epitrochoid.* It is evident that the latter is the general curve of which the cycloid is but a special case.

In describing an epitrochoid it is most convenient to associate it with the corresponding cycloid.

In Fig. 40 $A\ G$ is the diameter of the rolling circle whose generating *radius* is $A\ E$.

$A\ D\ B$ is the cycloid described by the point at the extremity of the radius. $A\ B =$ the circumference of the rolling circle, which is the base, $C\ D =$ the diameter of the rolling circle, the axis; A and B the cusps, and D the vertex. $I\ J\ K$ and $L\ M\ N$ are epitrochoid curves generated by the points I and L upon the generating radius.

THEOREM XVIII.

*The equation of the epitrochoid is $x =$ the arc whose ver. sin. is $y, - \sqrt{2ry - y^2} +$ sin. to generating point — sin. to secondary point.**

In the cycloid, Fig. 40, referred to the axes $A\ G$ and $A\ B$, let P be the position of the generating point at any time. Then, if I is the secondary point, and $V\ O = E\ I$, V is the position of the secondary point when the generating point is at P.

In the equation of the cycloid, the value of x, for the point P, is $A\ Q$. In the equation of the epitrochoid, $I\ J\ K$, the value of x, when P is at the same point, is $A\ R$, that is, $A\ Q + Q\ R$. But $Q\ R = S\ O - T\ O$. $S\ O$ is the sin. $P\ O\ W$ to radius $P\ O$ and $T\ O$ is sin. $P\ O\ W$ to radius $V\ O$. Also from equation of cycloid $A\ Q =$ arc whose ver. sin. is $y, - \sqrt{2ry - y^2}$, hence $A\ R$, the abscissa of the point in the epitrochoid corresponding to the position of the generating point, or $x =$ arc whose ver.

*This form of expression is preserved in order to harmonize with the equation of the cycloid. By sin. to generating point is meant sin. of the arc of revolution $P\ W$, determined by the point of contact and the generating point, that is, the sin. of the angle $P\ O\ W$ to radius $P\ O$. By sin. to secondary point is meant sin. of angle $P\ O\ W$ to radius $V\ O$.

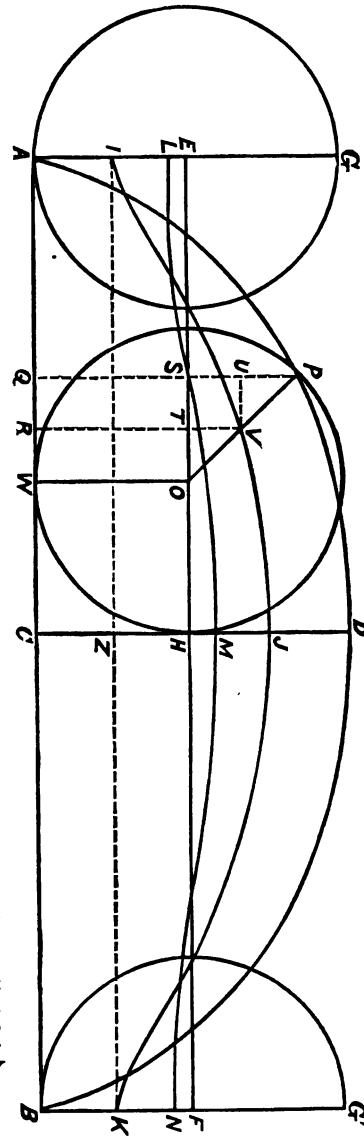


Fig. 40.

sin. is y, $-\sqrt{2ry-y^2}+$ sin. to generating point $-$ sin. to secondary point*.

When x in the equation of the epitrochoid $=0$, the terms of the second member are severally and collectively 0.

When the arc described by the generating radius is 90°, $x=$ quadrant of the rolling circle $-$ distance of secondary point from the centre, since $\sqrt{2ry-y^2}$ and sin. generating point are each equal to r, and sin. secondary point is its distance from the centre.

When the arc $=180°$, $x=$ semi-circumference of the rolling circle, which applies to all points in the generating radius, for $y=2r$, $\sqrt{2ry-y^2}=0$ and sin. generating point, and sin. secondary point also $=0$.

When arc $=270°$, $x=\frac{3}{4}$ circumference of the rolling circle $+$ distance secondary point.

When arc $=360°$, $x=$ circumference of the rolling circle. To obtain these latter results it must be observed, that, when $y=r$ the value of $\sqrt{2ry-y^2}$ is $\pm r$, and also that sin. generating point and secondary point may be positive or negative for the same value of y. This reduces the general equation, when $y=r$, to the form, $x=$ arc whose ver. sin. is $r-\pm r\pm r\pm r'$, r' being the distance of the secondary point from the centre of the rolling circle. The selection of signs must agree with the rule for signs of circular functions.

$r-+r+r-r'$, being used for the first and second quadrants, and $r--r-r+r'$ for the third and fourth quadrants.

The cycloid together with all the epitrochoid curves are symmetrical with reference to the axis, but unsymmetrical with reference to the line traced by the centre of the rolling circle. As the secondary point approaches the centre, however, the curve approaches symmetry and coincidence with the straight line EF.

When the secondary point is in the centre, the equation which

*It must be observed that y in this equation is the ordinate to the generating point, and not the ordinate to the point in the epitrochoid. The latter if desired may readily be obtained. This is the general equation of the cycloidal curve, of which the old equation is but the special case, in which the generating point and the secondary point coincide, whereby the addition to the old equation vanishes. It may be convenient to regard the radius of the rolling circle as unity, and the distance of the secondary point from the centre as a fraction. The terms sin. and ver. sin., in the general equation, pertain to the same angle, differing only in radii.

applies to the path pursued, is $x =$ the arc, whose ver. sin. is y, for sin. secondary point $= 0$ and sin. generating point $= \sqrt{2ry - y^2}$

The terms, $\sqrt{2ry - y^2}$ and sin. generating point, being synonymous, if (Fig. 40) $PO = m.\ VO$, the abbreviated form of the equation of the epitrochoid is $x =$ *arc whose ver. sin. is* $y, - \frac{\sqrt{2ry - y^2}}{m}$, or $x =$ arc whose ver. sin. y — sin. secondary point.

To Find the Area of the Epitrochoid.

The area of the epitrochoid is the surface (Fig. 40) included within the curve IJK and the base IZK.

In any position of the rolling circle it is evident that as the circle advances by an indefinitely small distance, the area described by the generating radius of the epitrochoid, PO, is to the area described by the generating radius of the epitrochoid, VO. As PO^2 is to VO^2, that is,

$$\text{area } PO : \text{area } VO :: PO^2 : VO^2;$$

hence, the sums of all the areas described by PO and VO respectively, are as $PO^2 : VO^2$, *i. e.*, total areas PO : total areas VO $:: PO^2 : VO^2$; $\therefore$ total areas $VO = VO^2 \times$ total area $PO \div PO^2$, $\therefore$ area epitrochoid $= \frac{\text{three times area gen. circle} \times VO^2}{PO^2}$.

When VO is 0, area epitrochoid is 0.
When $VO = PO$ " " = cycloid.
When $VO = \frac{1}{2}PO$ " " = $\frac{1}{4}$ cycloid.

The epicycloid is the curve traced by the generating point when the rolling circle moves upon the outside of another circle, the hypocycloid when the rolling circle moves upon the inside of the fixed circle. The relation of the cycloid to the epicycloid depends upon the relation of the rolling to the fixed circle. All cycloids are similar, while epicycloids are similar only when they cover the same number of degrees in the fixed circle.

Newton rightly finds the area of the epicycloid $\left(3 + \frac{2b}{a}\right), \times$ area of the rolling circle, in which b is the radius of the rolling and a the radius of the fixed circle.

Since $3 \times$ area of rolling circle is the area of the cycloid, $\frac{2b}{a} \times$

area of rolling circle represents the difference between the cycloid and the epicycloid generated by the same rolling circle.

If $a = b$, the area of the epicycloid is five times that of the rolling circle, and the epicycloid covers the entire circumference of the fixed circle. It may readily be shown, that the sum of the areas of any number of epicycloids covering the circumference of the fixed circle, is equal to the area of the same number of cycloids, by the same rolling circle plus twice the area of the rolling circle. Generally, if $a = b\,m$, $\frac{2}{m} \times$ area of the rolling circle is the difference between the cycloid and the epicycloid.

The relations existing between the cycloid and epicycloid pertain as well to the relation of the epitrochoid of the cycloid and to the epitrochoid of the epicycloid.

The epitrochoid to the epicycloid represents the motion of the moon and other secondaries with reference to the sun and primaries.

New Method of Determining the Area of the Cycloid, the Epicycloid and the Hypocycloid.

The area described by the generating radius depends: 1st, upon the revolution of the rolling circle; 2nd, upon the translation of the centre; and 3rd, upon the curvature of the base.

The area described by the generating radius depending upon the revolution of the rolling circle is the *area* of the rolling circle. The additional area, depending upon the translation of the centre, is the area described by the chord of the rolling circle which connects the generating point with the point of contact, as $B\,C$ Fig. 41.

Designate the points C and D, being so situated that the arc $B\,C \times$ the arc $B\,D = 180°$, as supplementary points, and let C and D be any such points in the circumference of the rolling circle.

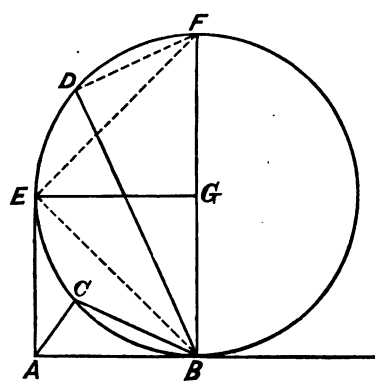


Fig. 41.

The additional area depending upon the translation of the centre varies as the square of the connecting chord. Now, whatever the positions of the supplementary points C and D, $B\,C^2 + B\,D^2 = B\,F^2$

$= 2\,BE^2$; that is, the sum of the areas by any supplementary chords is a constant quantity and equal to twice the area by the chord of 90°; hence, the average area by the connecting chord is equal to the area by the chord BE; $\because$ the additional area depending on the translation of the centre, in an entire revolution of the rolling circle, is the area of a circle whose radius is the chord of 90° of the rolling circle, which is readily seen to be twice the area of the rolling circle. The entire area then of the cycloid is the area of rolling circle, depending upon its revolution, plus twice the area of the rolling circle, depending on translation, or three times the area of the rolling circle.

The increment to the area of the cycloid at any point varies as $BG^2 + BC^2$.

The area of the epicycloid depends on the farther increase occasioned by the curvature of the base. The increment at any point is determined by the connecting chord and the ration of the radii of the circles. Supposing the fixed and the rolling circles equal, one epicycloid will cover the entire circumference of the fixed circle, hence the connecting chord in addition to its revolution described in the cycloid, has described another revolution owing to the curvature of the base. This fact is very readily proven by the method just applied to the cycloid. The area of the epicycloid then, when the circles are equal, is the area of the rolling circle depending on its revolution, plus twice the area of the rolling circle, depending on the translation of the centre, plus twice the area of the rolling circle depending on the curvature of the base, or five times the area of the rolling circle. If the radius of the fixed circle is twice that of the rolling, the epicycloid will cover one-half of the circumference of the fixed circle, whence the additional area caused by curvature of the base is only half as great as when the circles are equal; that is, the area of the epicycloid is four times that of the rolling circle. If the radius of the fixed circle is five times that of the rolling, the increase in five epicycloids, depending on curvature, is twice the area of the rolling circle; hence the increase in each epicycloid is $\frac{1}{5}$ of twice or $\frac{2}{5}$ of the area of the rolling circle. Generally, if a represents the radius of the fixed, and b that of the rolling circle, $\frac{a}{b}$ is the number of epicycloids which cover the fixed circle, and the increase over the cycloid belonging

to each epicycloid is $2 \div \frac{a}{b} = \frac{2b}{a}$ times the area of the rolling circle; $\therefore$ the area of the epicycloid is $\left(3 + \frac{2b}{a}\right)$ times the area of the rolling circle, which agrees with the result obtained by Newton. When a is infinite, or b is zero, the value of the cycloid and epicycloid is the same.

By the same method it is shown that the area of the hypocycloid is $\left(3 - \frac{2b}{a}\right)$ times the area of the rolling circle. When $a = b$ the area of the hypocycloid is the area of the rolling circle, which is evident, since there is no translation of the centre. When $a = 2b$, the area of the hypocycloid is twice that of the rolling circle, and the hypocycloid itself is the diameter of the fixed circle, a straight line. When a is infinity, or b is zero, the hypocycloid becomes a cycloid.

Aided by these discussions upon the cycloid, which are of course not all essential to the purpose, we are now prepared to construct a curve illustrative of secondary motion about a central force.

Since a figure drawn upon a small scale must exaggerate minor features in order to render them conspicuous we will attempt an illustration with an imaginary figure. Conceive a fixed and a rolling circle, the latter upon the convex side of the former of such dimensions that the radius of the fixed is about thirteen times that of the rolling circle. If the sum of these radii expresses the distance of the earth from the centre of its orbit, the path of the centre of the rolling circle represents the true orbit of the earth. Upon the radius of the rolling circle conceive a point whose distance from the centre is about $\frac{1}{80}$ of the radius. This point nearly represents the distance of the moon from the earth. The locus of this point, as the rolling circle performs its 13 revolutions upon the fixed circle, represents with considerable accuracy the path of the moon with reference to the earth and the sun. The path thus described by the given point is an epitrochoid of the epicycloid. Owing to the eccentricity of its orbit and solar disturbance the path of the moon is sensibly modified.

The distance of the moon from the earth might, with more security, be diminished than increased. For instance, if its distance were nine times as great as now (which is not equal to that of

Saturn's remotest satellite), its hourly velocity in the orbit about the earth should be but $\frac{1}{3}$ as great as now; and, since its orbit would be nine times as great in circumference, its period of revolution would be twenty-seven times as long as now; about once in two years. This condition would expose the moon for so long a period to the sun's influence that it could not possibly resume its course. It might readily be determined by means of the foregoing demonstrations at what distance from the earth the precipitation of the moon upon the sun would be inevitable. In general it is required that the satellites of the nearer planets nestle much closer to their primaries than those of the remoter planets. Neptune might command a satellite many millions of miles away, the influence of the sun in that quarter being but $\frac{1}{900}$ as intense as upon the moon, and the attraction of Neptune being about ninety times as great as that of the earth, aggregating an advantage in favor of Neptune over the earth in its control of a satellite of 1 to 80,000. On the other hand, if Mercury is associated with a satellite, its distance must be comparatively small, since, the influence of the sun is nine times as great as upon the earth, while the mass of Mercury is but $\frac{1}{9}$th, making a disadvantage to Mercury as compared with the earth of 81 to 1. At a distance of 3,000 miles from Mercury, performing a revolution in about 20 hours, a satellite to Mercury would suffer more violence from the sun than does the moon in its present position. Neptune might with equal success regulate the movement of a satellite at the distance of 50,000,000 miles. The period of such satellite to Neptune would be over 100 years.

Secondary orbits are greatly modified by the *unevenness* of the disturbances arising from the attraction of the sun. It is possible to determine the amount and the effect of such unevenness, and also, to tell what would be the behaviour of a satellite, if the variations of intensity in the sun's action did not affect it.

It would not be necessary that any dispute should arise concerning the effect of the sun's disturbing action upon a satellite, for the same explanation that satisfies the elliptical theory correctly satisfies also the circular. It so happens, however, that though certain conclusions in Loomis' Treatise may be correct they are not logical—that though certain effects may be correctly stated, they are not, owing to the causes to which they have been assigned. In order to be clearly understood it is best to reproduce from Loomis' Treatise of Astronomy, p. 144, Art. 268.

"GENERAL EFFECT OF THE SUN'S DISTURBING ACTION.

Let us suppose that the projectile motions of the earth and moon are destroyed, and that they are allowed to fall freely toward the sun. If the moon was in conjunction with the sun, it would be more attracted than the earth, and fall with greater velocity toward the sun, so that *the distance of the moon from the earth would be increased in the fall.* If the moon was in opposition she would be less attracted than the earth by the sun, and would fall with a less velocity toward the sun than the earth, and the moon would be left behind by the earth, so that the distance of the moon from the earth would be *increased in this case also.* If the moon was in one of the quarters, then the earth and the moon being both attracted toward the centre of the sun, would approach the sun, and at the same time would necessarily approach each other, so that their distance from each other would in this case be diminished. Now, whenever the action of the sun would *increase* their distance, if they were allowed to fall toward the sun, it produces the same effect as if their gravity to each other was diminished; and, whenever the action of the sun would diminish their distance, their gravity to each other is increased. Hence, we conclude that *the sun's action increases the gravity of the moon to the earth at the quadrature, and diminishes it at the syzygies.*"

The great mistake committed in this article is, that, while projectile motion is suspended, no allowance is made for the attraction between the earth and moon. If attraction were not mutual—in other words, if the sun attracted the earth and the moon, while they did not attract each other, the statements in Art. 268 would be correct; but, as it is, they are mischievously erroneous. For, suppose the moon in conjunction with the earth when the projectile force is suspended. Say, the attraction of the sun equals $2\frac{1}{5}$ times that of the earth; then, since the sun and the earth draw upon the moon in opposite directions, the moon will advance in the direction of the greater force by the difference of their intensities; that is, the moon will advance toward the sun by a force represented by $1\frac{1}{5}$ times the attraction of the earth. The earth, however, will be drawn in the direction of the sun by a force compounded of the whole attraction of the sun and the moon, which will aggregate over $2\frac{1}{5}$ times the effect of the earth's attraction. Hence the earth will approach the sun twice as rapidly as the moon and will soon overtake it. In

other words, while the earth and the moon are approaching the sun, they approach each other with about equal rapidity, so that instead of their distance from each other being increased in the fall, they will come in contact with each other in the early part of their flight. The same line of argument will prove that if projectile motion be suspended, when the moon is in opposition, the moon will soon overtake the earth instead of falling farther and farther behind, as taught in Art. 268. Should projectile motion be suspended when the moon is at quadrature, Art. 268 says rightfully that the earth and moon will approach each other as they approach the sun. According to the teaching of this article there must be a point somewhere in the quadrant, at which, if projectile motion be suspended, the earth and moon will maintain their relative positions unchanged during their entire approach to the sun. So far the statements contained in this article are at variance with fact, owing to a disregard of the law of mutuality.

As for the conclusion that the sun's action increases the gravity of the moon to the earth at quadrature and diminishes at syzygy, it is true, but not as a consequence to what precedes in this same article. A close examination into the conduct of the earth and moon, in the cases proposed in this article, it will be found that in their approach upon the sun from conjunction or opposition the effect of the sun's action is to retard the approach of the moon to the earth, which is in effect the same as diminishing the gravity of the moon. At quadrature the effect of the sun's action will be to hasten the contact, which is in effect the same as to increase the gravity of the moon. There are points in the quadrants at which the distance of the moon from the sun is the same as the distance of the earth from the sun; hence, they will approach each other with the same freedom as though the sun did not attract either. We conclude then that, let the projectile force of the earth and moon be suspended in any of their relative positions, they will approach each other, and that the effect of the sun's attraction is to *hasten* their approach at quadrature, and to *retard* the same at syzygies, from which we conclude that the action of the sun increases the gravity of the moon to the earth at quadratures and diminishes it at the syzygies.

The variations of lunar motion arising from the action of the sun, are chiefly owing to the *differences* in the *distance* of the moon from

the sun. Should the secondary motion of the moon be suspended at quadrature, and the attraction between the earth and moon cease at the same time, while they pursue their way in the same orbit about the sun, some 240,000 miles apart, the action of the sun would not alter the distance between them. If now the moon were suddenly, with this same velocity, placed in the position of conjunction, its velocity would be inadequate to maintain its distance from the sun; hence, it would be drawn away from the earth and describe an orbit about the sun not concentric with the orbit of the earth. From the formulæ pertaining to the elements of orbits, it is s hown, that the velocity necessary to similar orbits varies inversely as the square root of the distance; hence, the nearer the sun, the greater the required velocity, and *vice versa*. For this reason the moon is drawn toward the sun in the case just supposed. And, for the same reason, the moon would, if similarly placed at opposition, tend to recede from the sun; and, of course, at the same time from the earth. Hence we conclude that *the tendency of the sun's disturbance is to elongate the orbit of the moon in the line passing through the earth and the sun.*

The particles of matter upon and near the surface of the earth's equatorial regions, are subject to a varying tendency to the moon, even greater than the variations in the moon's tendency to the sun. Each particle may be regarded as a satellite moving about the centre of the earth, subject to disturbance from the moon. The difference in distance from the moon, between two particles on opposite sides of the earth, is about $\frac{1}{80}$ of the moon's distance from the earth, by reason of which the tendency to the moon on the near side of the earth is about $\frac{1}{10}$ greater than the tendency on the opposite side. Midway between these points the tendency to the moon is a mean. The result of these tendencies is a bulge on the near and opposite side of the earth, in the waters of the ocean known as the tidal wave.

The existing theory of tides, if we know it aright, is that the waters on the side near the moon being attracted most forcibly, they are lifted above the general level. On the opposite side, the water being more distant than the main mass of the earth, they are left behind, causing a similar wave. As auxiliary causes in the formation of tidal waves, we briefly offer the following: In the course of a daily revolution of the earth, the particles of which the earth is constituted, alternately approach and recede from the moon. In

order to satisfy an evident analogy, let us call the point near the moon conjunction and the other side opposition. As a particle of matter passes from opposition to conjunction the attraction of the moon urges it onward, causing it, if free to move, to travel more rapidly than the main mass of the earth. From conjunction to opposition the order is reversed, on account of which the accelerated and the retarded particles crowd upon each other at conjunction, causing the tidal swell.

On the opposite side of the earth the similar tide is owing, to a great extent, to the fact that each particle is held in position by less than the mean of attraction, which increases its tendency to withdraw from the moon, and, of necessity, from the centre of the earth.

It is not the purpose at present to discuss the subject of tidal waves any farther than pertains to the principles of central force just alluded to. It is confidently believed that planetary motion and the tidal wave are owing to a common cause.

The meditations which gave rise to the foregoing explanation of the tidal wave, together with the discussions belonging to the cycloid and other closely related curves, suggested to the author a general theory of waves as noticed upon the surface of disturbed waters. An outline of the theory is all that will be offered at present, since the subject has no vital connection with the general intent of this work, and also because the author has not satisfied himself by evidences.

THE THEORY.

The *distance* between the crests of two successive waves depends upon the *depth* of agitation. The *depth* of agitation is to the distance between the crests as the radius to the circumference of a circle. The elevation of the crest depends upon the *intensity* of the agitation, displaying the various forms of the epitrochoid whose limit is the cycloid.

The cycloid is the curvature of the wave depression at the limit which causes the crest to burst into spray. The profusion of spray depends upon the farther increase of the disturbing cause, which may be an addition of foreign force, or an arrest of movement caused by the shallowing of waters. The latter cause accounts for the increased violence and the crowding tendency of the breakers of the beach.

The author believes that the various demonstrations contained in

these pages pertaining to the action of central forces, if not in themselves a solution, are the much needed helps to a successful solution of the *problem of three bodies.*

These pages are now submitted to the public for unbiased consideration. The author will not hesitate to avow that he has endeavored in this work to avoid the spirit of iconoclasm, ever keeping in view the higher purpose of correcting, simplifying and enlarging the scope of science. The reader will be left to judge whether the author has succeeded in maintaining the declarations contained in the following public announcement of Sept. 9, 1886, furnished by the *Lancaster Intelligencer* of that date, and to which our own heading was,

A SCIENTIFIC REVOLUTION.

The progress of intellectual development is in no instance more evident than in the advancement of astronomical science; and no other science is so well calculated to enlarge our conception of the immensity, beauty and harmonious stability of the handicraft of God. Ages upon ages the phenomena of the heavens were watched and interpreted with superstitious fear. For fourteen centuries more, dating almost with the commencement of the Christian era, the now absurd notion prevailed that the earth is the centre of the universe, about which the myriad hosts of heavenly bodies perform their daily course. Examined in the light of advancing science the movements of the planets challenged the correctness of this belief, and invited investigation. It seems strange that in spite of the renowned intellects of the heathen ages, the solution of so conspicuous a problem should remain for so late a period as the sixteenth century.

Copernicus, though greatly assisted by the doctrine of certain Greek writers, may truly be regarded as the father of modern astronomy. By him we derive the plan of the solar system, that the sun is the centre about which the planets revolve. Somewhat later, Kepler furnishes the law of equal areas in equal times. But greater than these and older than these, though later discovered, is Newton's law of universal gravitation, by the aid of which the mysteries of heaven may be uncovered millions beyond the limit of the best telescope.

The natural order in the development of mathematical astronomy is that the law of *gravitation* should precede. Kepler's law and the system of Copernicus are but the natural offspring of universal gravitation. By reason of the inverse order of development, grave errors have been committed and perpetuated to the present, much to the embarrassment of science, chief among which is the doctrine that the heavenly bodies move in ellipses, parabolas and hyperbolas.

Kepler in his laws announces that the orbits of the planets are elliptical and that the sun occupies one of the foci. Newton accepts and endeavors to demonstrate the correctness and enlarges by showing the possibility in the case of great velocities of parabolic and hyperbolic orbits, in which comets which are not periodic are supposed to move. All of this it is proposed by demonstration to prove incorrect.

The challenge then, which is extended to all who choose to accept it, is, that the laws of Kepler pertaining to the form of orbits and the relation of the periodic time to the mean distance together with all of Newton's demonstrations and enlargements upon the same are faulty. It is observed, then, that only one of the three Kepler laws is accepted.

A brief synopsis of what it is proposed to give in exchange for the adopted doctrine is all that the limit of a newspaper article will admit.

It will be shown by *demonstration*, and not by *observation*, as Kepler did, that the form of all orbits is *circular* and that the central force may occupy any position in the orbit from centre to circumference, excluding only *the* circumference. That the position of the central force with reference to the centre of the orbit—in other words, that the eccentricity of an orbit depends upon the relation between the centrifugal and centripetal forces of which it (the eccentricity) is the direct and exact measure; that the *degree* of *eccentricity* determines the amount of *surplus centrifugal force* at the *perihelion* of an orbit.

Using what is called the non-eccentric orbit, in which the central force occupies the centre of the orbit, as the unit of comparison, the law and formulæ are derived which establish the relation among the elements of any orbit: viz., the velocity, area, period and eccentricity.

It will also be shown by these demonstrations that if any planet were deprived of about one-fourth of its centrifugal momentum the consequence in one-half of a revolution would be precipitation into the sun, which is also radically opposed to Newton, who does not concede the possibility of precipitation so long as a remnant of centrifugal force remains.

Conscious that the burden of proof and disproof rests upon the author, he is prepared to show where Newton, who accepted the correctness of Kepler by mistaking a *constant* for a *variable*, passed within a hair's breadth of the truth, and then by violation of an axiom arrived at conclusions upon the merits of which rests the present doctrine of orbital motion; the merits of the ellipse, the parabola and the hyperbola, as applied by Newton to orbits, all rest upon a common foundation and must share a common fate. It will be shown that the only condition in which the law of gravitation applies to elliptical motion is when both foci are occupied by equal

forces, and that only as a special case, which again disqualifies its application to the solar system.

By a generalization of the cycloidal equation, which represents the *path* of *any point* upon the *radius* of the generating circle, the absolute and relative motions of a primary and its secondary about the sun is expressed in general terms which satisfies the conditions of the Problem of Three Bodies.

The announcement that the orbits of the comets even are circular, may to the educated appear unworthy of notice; but when we remember that so far as *observation* is concerned, it is impossible to distinguish between small sections of conic curves described upon a large scale we should at least be open to conviction in the face of satisfactory demonstration.

With this announcement as a forerunner, the author is prepared in the near future to present to the public the demonstrations by which he has attained to these and various other significant conclusions.

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New Danville, Pa.

APPENDIX.

Article I.

As observed (p. 17) it has been objected publicly by several professors of mathematics that $P\ V$ can not be consistently omitted from the expression of variation as used by Newton. Their objection is based on the evidence that $P\ V$ does not represent a *constant*; and charge, moreover, that we inconsistently recognize it, sometimes as *constant*, at others *variable*. Admitting that by one utterance we pronounce Newton's chord $P\ V$ as constant, and by another a variable, we defend by contending that, though it may appear paradoxical, such is the real nature of the quantity. *Variable-constant* is the most appropriate name for such values. The chord $P\ V$ is variable in the general aspect because its value changes with every change of value assigned to the radius-vector. It is *constant* in the *special* aspect (which is the only aspect which renders demonstration possible), because it is a *chord passing through two fixed points*, designated by the actual positions of the central force and the body moving in the given orbit. It is in this latter sense, clearly, that Newton uses $P\ V$ in his demonstration and upon the reasonable evidence of its constancy when so used we demand its removal from the expression of variation.

We beg to enlarge upon our vindication because it requires double evidence to disprove an error so long entertained as truth.

Quantities, as generally distinguished by algebraists, are either *constant* or *variable*. There are, however, three kinds illustrated by 4, a and x, of which 4 is the *constant*, x the *variable*, and a, partaking of the nature of both, the *variable-constant*. In all algebraic processes the *constant* 4 and the *variable-constant* a are treated alike as *integral constants*.

Without entering upon an exhaustive discussion on the point, we invite an examination of the rules and cases of *variation* as treated by any standard authority. It will be found that none but *independent variables* are retained in the expression. Examples might readily be furnished in proof, but we think suggestion sufficient.

To show more clearly what we mean by a variable constant let us examine the equation of a straight line passing through the origin of co-ordinates $y = ax$. Since a represents the tangent of the angle, formed by the intersection of the line with the axis of abscissas, it is evident that an infinite variety of values may be assigned to a as well as to x or y; that a is the most *independently variable* factor of the equation; and that the possibility of determining any *particular* line depends upon the assigning of a particular (constant) value to a. We say of the co-ordinates of the straight line, that $x \propto y$, which is true of a *particular* line, but not of the *general* expression $y = ax$, for then $y \propto ax$. The reason for omitting a from the expression of variation, whenever a particular value is assigned to it, is identical to the reason that the value of the chord PV should be omitted in Newton's demonstration, namely, that by assigning positions to two points of a chord the chord is determined—*constant*. In harmony with the conception of a *variable-constant* we reassert that the various values of Newton's PV are best represented as an *infinite number of constant chords*.

By reference to Fig. 9, p. 16 it will be seen that SP and PV are relatively dependent, and that the value of either determines the other. For, by properties of the circle, whatever the position of P in the circumference, $SP(PV - SP) = SD \times Sd$, the product of the parts into which the position of the central force divides the diameter passing through it. Designating the constant $SD \times Sd$ by a, we derive $SP(PV - SP) = a$.

Solving this equation, regarding SP or the radius-vector as independent variable we find $SP = \frac{PV}{2} \pm \sqrt{\frac{PV^2}{4} - a}$; which is strictly general and shows that PV *depends* for its value on SP. Regarding PV as the independent variable in $SP(PV - SP) = a$, we find $PV = \frac{a}{SP} + SP$.

Either SP or PV might be regarded as an independent variable, but certainly not both. For many reasons, it is most proper to regard SP as the independent variable, whence Newton's form of variation reduces to $F \propto \frac{1}{SP^2}$, which is in harmony with the new.

Article II.

The demonstration of Theorem VIII, p. 25, might be conducted with possibly greater simplicity without resort to trigonometry, as follows,

$$2\,R - A\,C : P\,C :: P\,C : A\,C.$$

When the angle $P\,B\,A$ is infinitely small $A\,C$ may be omitted from the first term,

$$\text{whence } 2\,R : P\,C :: P\,C : A\,C,$$

$$\text{and } 2\,A\,C = \frac{P\,C^2}{R}.$$

From principles of mechanics the accelerating effect of attraction (in this case $A\,C) = \frac{1}{2}\,F\,T^2$, being the product of the force by the square of the time. The time being unity, we have

$$F = 2\,A\,C = \frac{P\,C^2}{R}.$$

Let r represent the radius-vector and H the area described in a unit of time. When the unit of time is taken so small that the arc described is *infinitely* small, the perpendicular $P\,C =$ the arc $A\,P$; hence H which equals $\frac{1}{2}\,r \times$ arc $A\,P =$ also $\frac{1}{2}\,r \times$ the perpendicular $P\,C \therefore P\,C = \frac{2\,H}{r}$ and $P\,C^2 = \frac{4\,H^2}{r^2}$. Substituting this value of $P\,C^2$ in the equation $F = \frac{P\,C^2}{R}$, we derive $F = \frac{4\,H^2}{R\,r^2}$.

Since R and the numerator are known $F \propto \frac{1}{r^2}$, which is the proposition.

Article III.

A body moves in an equiangular spiral, determine the law of force tending to the pole.

Newton solves this problem (Prop. IX) and finds the right result. We offer the following method which is original with us :

Let $S\,A\,B$ and $S\,B\,C$ be small sections of an equiangular spiral, whose pole is at S and the angles $A\,S\,B$ and $B\,S\,D$ equal.

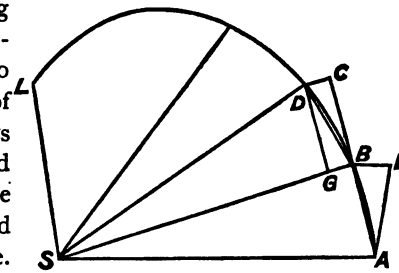

Fig. 42.

At A suppose the moving body impelled by a centrifugal impulse sufficient to carry it to E in a unit of time, while $B\ E$ represents the centripetal impulse toward the pole, then will B be the place of the body at the end of the given interval of time. If the body at B were free to move without any disturbance from S, it would move in the direction $B\ C$. But since by condition of the problem the body at the end of another interval of time (either equal or unequal) is in the curve at D, then $C\ D$ parallel to $B\ S$ must represent the effect of the centripetal force.

Now if $A\ B$ and $B\ D$ be taken infinitely small they will be arcs of the curve. Then, from the nature of the cvrve, it is evident that the angle $S\ B\ A$ = angle $S\ D\ B$; also, since $A\ C$ is a straight line and $C\ D$ parallel to $B\ S$, the angle $S\ B\ A$ = angle $D\ C\ B$, hence the angle $S\ D\ B$ = angle $D\ C\ B$. The angle $S\ B\ D$ being also equal to $C\ B\ D$, the triangles $S\ D\ B$ and $D\ C\ B$ are similar,

$$\therefore\ C\ D : D\ B :: D\ B : B\ S,$$

$$D\ B = \frac{D\ B^2}{S\ P}.$$

But $C\ B = \frac{1}{2} F t^2$; whence $F = \dfrac{2\ D\ B^2}{S\ P\ t^2}$.

If H be the area described by the radius-vector in a unit of time and t the time of traversing $B\ D$, then $H\ t$, the area of $S\ B\ D = \frac{1}{2} S\ B \times D\ G$, perpendicular to $S\ B$. Ultimately $D\ G = D\ B$, hence ultimately, $H\ t = \frac{1}{2} S\ B.\ D\ B$, whence $H^2 = \dfrac{S B^2\ D B^2}{4\ t^2}$.

$$\therefore\ H^2 \times \frac{8}{S\ B^3} = \frac{S\ B^2\ D\ B^2}{4\ t^2} \times \frac{8}{S\ B^3} = \frac{2\ D\ B^2}{S\ B\ t^2} = F.$$

that is, $F = \dfrac{8\ H^2}{S B^3}$, whence $F \propto \dfrac{1}{S\ B^3}$, which is the law of force for the equiangular spiral.

To find the velocity of a body describing an equiangular spiral, subject to a force tending to the pole.

$V^2 = F \frac{1}{2} P V$ is the formula for the square of the velocity in any orbit, F being the accelerating effect of the central force and $P V$ the chord of curvature passing through the centre of force. In the equiangular spiral $P V = 2 S B$ whence

$$V^2 = \frac{8 H^2}{S B^3} \times S B = \frac{8 H^2}{S B^2} \therefore V = \frac{H}{S B} \sqrt{8}.$$

To find the time of describing any part of an equiangular spiral.

The area $S A L$ of which $S A$, $S L$ are the bounding radii, is $\frac{1}{4}$ $(S A^2 = S L^2)$ tan. a, a being the angle at which the curve cuts the radii, $H t = \frac{1}{4} (S A^2 - S L^2)$ tan. a.

$$t = \frac{(S A^2 - S L^2)}{4 H} \text{ tan. } a.$$

Although the law of inverse cube accounts for the motion of bodies in equiangular spirals, it must not be supposed that all bodies subject to a central force, varying inversely as the cube of the distance, will move in equiangular spirals. Indeed, the case is so special, that if there be any variation in the *velocity* of the moving body or the angle between centrifugal and centripetal tendencies, the path is no longer an equiangular spiral. It does not, therefore, follow that the law of inverse cube under different conditions may not pertain to other curves.

Article IV.

Having denied the correctness of Newton's conclusion, that, if a body moves in a parabola, the force tending to the focus varies inversely as the square of the distance, we offer the following demonstration, showing that if a moving body subject to a centre of force describes a *common* parabola, the accelerating effect of the central force varies inversely as the cube of the distance.

A body moves in the parabola A P G, determined by a central force in the focus S; determine the law of force.

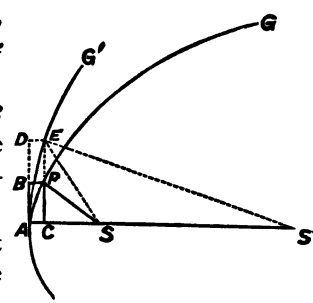

Fig. 43.

At A the centrifugal tendency $A B$ is perpendicular to the axis of the curve and to the direction of centripetal force.

Suppose the velocity such, that while the accelerating effect of the central force $= 2 A C$, $A B =$ tan-

gential progress; then will the body, at the end of the given period, be in P on the parabola.

From the general equation of the parabola

$y^2 = 2px$, we find,

$$PC^2 = 4AS, AC, \because AC = \frac{PC^2}{4AS}.$$

If H be the area of the sector (described in a unit of time) ASP, and the sector be taken infinitely small, whence the arc $AP =$ ordinate CP, we find,

$$H = AS\sqrt{AS.\ AC}.$$

$$H^2 = AS^3.\ AS \therefore AC = \frac{H^2}{AS^3} = \frac{H^2}{SP^3},$$

$$\text{and } F = 2AC = \frac{2H^2}{SP^3} \therefore F \propto \frac{1}{SP^3}.$$

In the case just considered, S represents the position of the centre of force and of the focus to the curve. Suppose now that while other conditions remain unchanged we double the velocity of the body moving from A. Then its position at the end of the given time is at E.

Assume that AEG' is also a parabolic curve, then, if, $y'^2 = 2p x'$ is the equation of P.

And $4y'^2 = 2p'x'$ is the equation of E, it is evident that $p \propto y^2$, that is, if AEG' is a parabola the distance from the apex to the focus AS' is four times AS. In general, if the velocity at A be multiplied by n the focal distance of the resulting parabola is increased n^2 times. Since $p \propto y^2$, and y represents velocity, it is manifest that $p \propto V^2$.

Regarding that condition, in which the centre of force is in the focus as a *non-eccentric* parabola, this problem of force may be managed with the same facility as circular motion; although, so far we know, these conditions are not met with in nature; hence no parabolic orbits.

Though other interesting problems might be framed in this connection, we dismiss the common parabola, asserting that its only solution as a problem of force is that which recognizes a central force, whose accelerating effect varies inversely as the cube of the distance, and, that said force must preside somewhere in the axis of the curve. It must be remembered that the possibility of parabolic motion allows just one particular velocity for the moving

body. Any increase beyond that velocity results in a pseudo-parabolic curve, whose limits are, on one side the true parabola, on the other, the axis of ordinates. Velocity less than that required in parabolic motion, results in a series of spirals whose limits are, on one side, the true parabola; on the other, the straight line from apex to focus. In the case of parabolic movement (though simply imaginary), there is, when the body leaves the apex, an excess of centrifugal over centripetal force. When, by the reduction of centrifugal force, as is the case in the spiral movement just alluded to, the central and tangential forces become equal, the body will describe a circle whose centre is the focus of the given parabola. In this case the circle is simply an equiangular spiral which cuts the radius-vector at 90°. As has been stated elsewhere, in the course of this treatise, the circle becomes a special case of movement under any law of central force, but in all cases except that of the inverse square the motion is non-eccentric. When the velocity with which the moving body leaves the apex, is greater than that required to produce circular motion, under the law of inverse cube, the moving body will be *constantly* receding from the central force because the accelerating tendency to the centre diminishes more rapidly than the momentum of the moving body. For the same reason, if the motion be less than that required for circular motion, the body moves in a spiral of precipitation within the limits of the circle.

Even this case of gradation from circular to parabolic motion finds no room, intermediately, for the ellipse, which, together with the hyperbola, we ignore as possible representations of orbital motion subject to a single centre of force, whatever be the *law* of that force. Unlike as the ellipse and the hyperbola may appear, their kinship is intimate as may readily be established. The former is a geometrical expression of the relations of *real* quantities, while the latter is the geometrical expression of relations of *imaginary* quantities. We might furnish evidences to the relief of this assertion, but they are too foreign to the task on hand.

If our comprehension of the various members of the family of parabolic curves, is not faulty, it may be shown by the same method as in Fig. 43, that if a body move in a cubical parabola subject to a centripetal force in the focus, that force varies inversely as the 4th power of the distance; if in a biquadratical parabola the acceler-

ating effect varies inversely as the 5th power; and generally if $y^n = ax$ is the equation of the curve described by a moving body subject only to central and tangential forces, the law of central force is $F \propto \frac{1}{y^n + 1}$. To show that this law of parabolic motion can not include the inverse square or the law of gravitation, we have only to observe, that if it were included, $n = 1$ whence $y^n = ax$ becomes $y = ax$, which is the equation of a straight line, and not a parabola. True, it may be regarded as the *limit* of the parabola in which the focus is in the apex.

Article V.

The mistake referred to on page vi, which the author takes liberty to correct at this late period, occurs in Theorem VII, p. 22. It is a minor proposition and does not, in the least, affect the general merit of this "new doctrine." The tendency of this error has been to affect a few other minor propositions, which we will rectify without repeating the demonstrations.

Prop. VII reads: *in any orbit the relation between centrifugal and centripetal force is constant.* This declaration stands in contradiction with the true relation between these forces, p. 31, viz: centripetal force varies as the 4th power of centrifugal force. Prop. VII should read: *In any orbit the ratio between the centripetal force and the angular progress of the radius-vector is constant.*

For, the area described by the radius-vector in an infinitely short period of time $= \frac{1}{2} R^2\, dt$, in which R is the radius-vector and dt the arc of its revolution described by a radius of unity. Now, this arc of revolution dt is the direct measure of the angular progress of the radius-vector, hence, the relation of the angular velocity to the radius-vector is the same as the relation of the arc dt to R.

Area in small unit of time $= \frac{1}{2} R^2\, dt$.

But, by the law of equal areas $\frac{1}{2} R^2\, dt$ is a constant for all values of R, hence $dt \propto \frac{1}{R^2}$.

But centripetal force $\propto \frac{1}{R^2}$, whence dt, and, therefore, the angular progress of the radius-vector $\propto$ centripetal force. $\because$ their ratio is constant, which is the proposition.

It will be observed that this correction of Prop. VII consists more in a change of terms than of conclusions. The mistake occurred, by supposing that the velocity of the extremity of the radius-

vector at any moment was the same as the velocity of the moving body at the same moment. The correct idea, and the one which should have been recognized in Prop. VII, is, that *the angular progress of the radius-vector of an orbit determines the apparent velocity of the moving body as viewed from the centre of force, while the real velocity of the moving body is measured by the angular progress of the radius of the orbit.*

Owing to the circumstance that, in Prop. XI, the radius-vector to the apex of the isosceles triangle is equal to the radius of the orbit, the progress of the vector at that point is the true measure, at once, of both *apparent* and *real* velocity.

It has been shown pp. 27–32, that, if, $A\,P\,C$, Fig. 18, represents a non-eccentric orbit about a central force at S, by doubling the centrifugal force at A, the result is an orbit whose radius is 4 times the radius of $A\,P\,C$. Let $A\,C' = 4$ times $A\,S$, then will $S\,P'' = 4\,S\,A$. By the law of equal areas, when the moving body is at P'' the *angular* velocity of the radius-vector is $\frac{1}{16}$ and the *true* velocity of the moving body $\frac{1}{4}$ as great as at A.

Hence the velocity at $P'' = \frac{1}{4}$ of 2 times the velocity belonging to the non-eccentric $A\,P\,C = \frac{1}{2}$ of its velocity. By transposition of the central force from S to C', when the moving member is at P'', the velocity of the moving body is exactly that required to maintain itself in the same orbit at a uniform velocity about the new centre of force, the orbit now being non-eccentric.

For we have seen (p. 31), and it is verified in the tabular statements of planetary velocities, that the hourly motion varies inversely as the square root of the distance. In the case just considered the distance $S\,P'' = 4\,A\,S$, whence the velocity at the distance $S\,P''$, necessary to sustain non-eccentric motion, is $\frac{1}{2}$ of the velocity needed for non-eccentric motion at the distance $S\,A$, which is the relation just found.

Theorem XII, in compliance with the fact that the progress of the extremity of the radius vector is not the true measure of actual velocity, should be somewhat modified as follows: *If a line drawn be perpendicular from the middle of the perihelion distance of an orbit, the arc described by the extremity of the radius vector when the moving body is upon that line, is equal, for all degrees of eccentricity, to the arc described by the extremity of the vector of the non-eccentric orbit in the same infinitely small space of time.* The discussion may

readily be harmonized with this change, but since the proposition is comparatively unimportant we leave it with this suggestion.

Finally, Theorem XIV requires reconsideration in sympathy with the change in Theorem VII, to do which more clearly, we reproduce Fig. 18, in which S represents the position of the central force and $A\,p\,S$ the semicircle of precipitation.

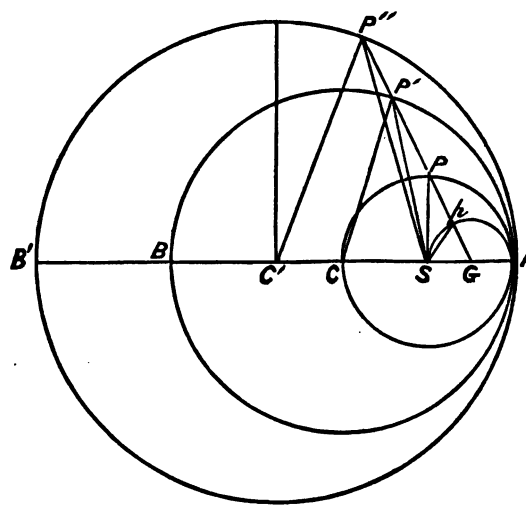

Fig. 18.

We have ascertained that the law of equal areas regulates the angular progress of the radius vector in the various positions of the moving body, while the consequent angular progress of the radius of the orbit determines the true velocity. Let us first find the relation of the angular progress of the radius vector to the angular progress of the radius in the semi-circle of precipitation $A\,p\,S$. At A, the motion of the body being at right angle with both radius $A\,G$ and vector $A\,S$, since the vector is twice the radius, the angular progress of the radius is twice that of the vector. Assume the point p *any* point in the semi-circle; then, since $S\,p\,G$ is an isosceles triangle it is evident that the angle $p\,S\,G$ at all times increases $\frac{1}{2}$ as fast as the angle $p\,G\,A$; that is, in all parts of the semi-circle of precipitation the angular velocity of the radius is twice as great as the angular velocity of the radius vector.

The vector moving in compliance with the law of equal areas, it follows that when the vector is $\frac{1}{n}$th as great, its angular velocity is